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Total positivity in loop groups, I: Whirls and curls

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Abstract

This is the first of a series of papers where we develop a theory of total positivity for loop groups. In this paper, we completely describe the totally nonnegative part of the polynomial loop group $GL_n(\mathbb{R}[t, t^{-1}])$, and for the formal loop group $GL_n(\mathbb{R}((t)))$ we describe the totally nonnegative points which are not totally positive. Furthermore, we make the connection with networks on the cylinder.

Our approach involves the introduction of distinguished generators, called whirls and curls, and we describe the commutation relations amongst them. These matrices play the same role as the poles and zeros of the Edrei–Thoma theorem classifying totally positive functions (corresponding to our case $n = 1$). We give a solution to the "factorization problem" using limits of ratios of minors. This is in a similar spirit to the Berenstein–Fomin–Zelevinsky Chamber Ansatz where ratios of minors are used. A birational symmetric group action arising in the commutation relation of curls appeared previously in Noumi–Yamada's study of discrete Painlevé dynamical systems and Berenstein–Kazhdan's study of geometric crystals.

Keywords: Total positivity; Loop groups

Contents

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1. Introduction

A matrix with real entries is *totally nonnegative* if all of its minors are nonnegative.

1.1. Total positivity in loop groups

Suppose $A(t)$ is a matrix with entries which are real polynomials, or real power series. When do we say that $A(t)$ is totally nonnegative? First associate to $A(t)$ an infinite periodic matrix X, as in the following example:

$$\underset{A(t)}{\begin{array}{c} \begin{pmatrix} 1+9t^2 & 2+5t \\ -1-2t-3t^2 & 8+3t-4t^2 \end{pmatrix} = \\ \begin{pmatrix} 1 & 2 \\ -1 & 8 \end{pmatrix} + t \begin{pmatrix} 0 & 5 \\ -2 & 3 \end{pmatrix} + t^2 \begin{pmatrix} 9 & 0 \\ -3 & 4 \end{pmatrix} \end{array}} \rightsquigarrow \underset{X}{\left(\begin{array}{c|cc|cc|cc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \hline \cdots & 0 & 5 & 9 & 0 & 0 & 0 & \cdots \\ \cdots & -2 & 3 & -3 & 4 & 0 & 0 & \cdots \\ \hline \cdots & 1 & 2 & 0 & 5 & 9 & 0 & \cdots \\ \cdots & -1 & 8 & -2 & 3 & -3 & 4 & \cdots \\ \hline \cdots & 0 & 0 & 1 & 2 & 0 & 5 & \cdots \\ \cdots & 0 & 0 & -1 & 8 & -2 & 3 & \cdots \\ \hline & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)}$$

We declare that $A(t)$ is totally nonnegative if and only if X is totally nonnegative. We use this to define and study the totally nonnegative part of the loop groups $GL_n(\mathbb{R}[t,t^{-1}])$ and $GL_n(\mathbb{R}((t)))$. Here $\mathbb{R}((t))$ denotes the field of formal Laurent series. We let $GL_n(\mathbb{R}((t)))_{\geq 0}$ denote the totally nonnegative part of $GL_n(\mathbb{R}((t)))$. Our main aim is to unify and generalize two classical subjects: total positivity in $GL_n(\mathbb{R})$ and totally positive functions.

1.2. Total positivity in $GL_n(\mathbb{R})$

The theory of totally positive matrices began in the 1930's in the works of Schoenberg [36] and Gantmacher–Krein [14] who discovered that totally positive matrices had remarkable spectral properties and a variation-diminishing property, cf. [18].

Let $e_i(a) \in GL_n(\mathbb{R})$ (resp. $f_i(a) \in GL_n(\mathbb{R})$) be the Chevalley generators, which differ from the identity matrix by a single entry in the i-th row (resp. column) equal to $a \in \mathbb{R}$ immediately above (resp. below) the diagonal. From our point of view, the most important classical result is:

Theorem 1.1 (*Loewner–Whitney Theorem [29,41]*)**.** *The space of non-singular totally nonnegative matrices $GL_n(\mathbb{R})_{\geq 0}$ is the multiplicative semigroup generated by Chevalley generators $e_i(a)$, $f_i(a)$ with positive parameters, and positive diagonal matrices.*

Theorem 1.1 led Lusztig [31] to his ground-breaking generalization of total positivity to reductive groups. Lusztig discovered deep connections between the theory of total positivity and his own theory of canonical bases in quantum groups [30]. In another direction, Fomin and Zelevinsky [12,11] studied the problem of parametrizing and testing for totally nonnegative matrices. Their attempt to classify the ways to test whether a matrix is totally nonnegative eventually led to the theory of *cluster algebras* [13].

Our first theorem (Theorem 2.6) establishes the analogue of Theorem 1.1 for the totally nonnegative part $GL_n(\mathbb{R}[t,t^{-1}])_{\geq 0}$ of the polynomial loop group, using the affine Chevalley generators. Note that the polynomial loop group itself is *not* generated by the torus and affine Chevalley generators with arbitrary parameters.

1.3. Totally positive functions

A formal power series $a(t) = 1 + a_1 t + a_2 t^2 + \cdots \in \mathbb{R}[[t]]$ can be considered a 1×1 matrix. We may then apply the definition of total nonnegativity in $GL_1(\mathbb{R}((t)))$ of Section 1.1 to define when a formal power series is totally nonnegative. Traditionally, formal power series $a(t)$ which are totally nonnegative are called *totally positive functions*. The coefficients $\{a_1, a_2, \ldots\}$ are said to form a *Polya frequency sequence*, see [5]. Totally positive functions were classified independently by Edrei and Thoma [9,39].

Theorem 1.2 (*Edrei–Thoma Theorem*). *Every totally positive function $a(t)$ has a unique expression as*

$$a(t) = e^{\gamma t} \frac{\prod_i (1 + \alpha_i t)}{\prod_i (1 - \beta_i t)},$$

where α_i, β_i and γ are nonnegative parameters satisfying $\alpha_1 \geq \alpha_2 \geq \cdots, \beta_1 \geq \beta_2 \geq \cdots$ and $\sum_i \alpha_i + \sum_i \beta_i < \infty$. In particular, totally positive functions are meromorphic functions, holomorphic in a neighborhood of 0.

Thoma [39] showed that the classification of totally positive functions was equivalent to the classification of characters of the infinite symmetric group S_∞. This connection was made more robust when Vershik and Kerov [40] interpreted the zeros and poles in Theorem 1.2 as asymptotic frequencies occurring in the representation theory of S_∞. No completely elementary proof of Theorem 1.2 seems to be known. For example, the original proofs of Edrei and Thoma use Nevanlinna theory from complex analysis, while Okounkov's proofs [33] rely on the connection with asymptotic representation theory.

One of the main themes of our work is the parallel between Theorems 1.1 and 1.2: $(1 + \alpha t)$, $1/(1 - \beta t)$, and $e^{\gamma t}$ can be thought of as semigroup generators for totally positive functions, when we also allow taking limits of products. We begin by considering the analogues of these generators for $n > 1$.

1.4. Whirls and curls

We introduce matrices $M(a_1, a_2, \ldots, a_n) \in GL_n(\mathbb{R}((t)))$ called *whirls*, and $N(b_1, b_2, \ldots, b_n) \in GL_n(\mathbb{R}((t)))$, called *curls*, depending on n real (usually nonnegative) parameters. For $n = 2$, their infinite periodic presentations look like

$$M(a_1, a_2) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & a_1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & a_2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & a_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & a_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$N(b_1, b_2) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & b_1 & b_1 b_2 & b_1^2 b_2 & b_1^2 b_2^2 & \cdots \\ \cdots & 0 & 1 & b_2 & b_1 b_2 & b_1 b_2^2 & \cdots \\ \cdots & 0 & 0 & 1 & b_1 & b_1 b_2 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & b_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Unlike Theorem 1.2, our theory is not commutative when $n > 1$. We study whirls and curls in detail. In Section 6, we describe the commutation relations for whirls and curls. In Section 9, we define the notion of infinite products of whirls or curls, and show (see Theorems 9.1, 9.5 and 9.6) the following.

Theorem (*Structure of Infinite Whirls and Curls*)**.** *Infinite products of whirls (or curls) form semigroups which are closed under multiplication by Chevalley generators on one side.*

1.5. The totally positive part $GL_n(\mathbb{R}((t)))_{>0}$

If X is an infinite periodic matrix corresponding to $A(t) \in GL_n(\mathbb{R}((t)))$, then every sufficiently southwest entry of X is necessarily equal to 0. Thus X is never totally positive in the usual sense, which requires all minors to be strictly positive. We define $A \in GL_n(\mathbb{R}((t)))_{\geq 0}$ to be *totally positive* if it is totally nonnegative, and in addition, all sufficiently northeast minors (see Section 2.2 for the precise definition) of the corresponding infinite periodic matrix are strictly positive. We show (Theorem 5.14):

Theorem (*Matrices of Finite Type*)**.** *The set $GL_n(\mathbb{R}((t)))_{\geq 0} - GL_n(\mathbb{R}((t)))_{>0}$ of totally nonnegative matrices in the formal loop group which are not totally positive is a semigroup generated by positive Chevalley generators, whirls, curls, shift matrices (defined in Section 4), and diagonal matrices.*

1.6. Canonical form

For simplicity, we restrict (using Theorem 4.2) to the subsemigroup $U_{\geq 0} \subset GL_n(\mathbb{R}((t)))_{\geq 0}$ consisting of matrices $A(t)$ with upper triangular infinite periodic representations. In Theorems 8.3 and 8.8, we establish a partial generalization of Theorem 1.2 to $n > 1$ (it is in fact a rather precise generalization of the result of Aissen et al. [1]). We call a matrix $Y \in U_{\geq 0}$ *entire* if all n^2 matrix entries are entire functions. The following results are our main theorems.

Theorem (*Canonical Form I*)**.** *Every $X \in U_{\geq 0}$ has a unique factorization as $X = Z\exp(Y)W$, where Z is a (possibly infinite) product of curls, W is a (possibly infinite) product of whirls, and Y is entire such that* $\exp(Y) \in U_{\geq 0}$.

The "limits of products" A and B in the following theorem are not necessarily single infinite products.

Theorem (*Canonical Form II*)**.** *Every matrix* $\exp(Y) \in U_{\geq 0}$ *with Y entire, has a factorization as* $\exp(Y) = AVB$, *where A and B are both limits of products of Chevalley generators, and $V \in U_{\geq 0}$ is regular.*

In [21], we strengthen this result by showing that the matrices A, V, B in the above theorem are unique. The notion of regular totally nonnegative matrices is introduced and discussed in Section 8. These results establish that every $X \in U_{\geq 0}$ has three "components": (a) a whirl and curl component, (b) a component consisting of products of Chevalley generators, and (c) a regular totally nonnegative matrix. We study (a) in detail here, but leave (b) and (c) for subsequent papers [21,22].

1.7. From planar networks to cylindric networks

A fundamental property of totally positive matrices is their realizability by planar weighted networks, connecting total positivity with combinatorics. By the Lindström theorem [28] and Theorem 1.1 (see also [6]) a matrix $X \in GL_n(\mathbb{R})$ is totally nonnegative if and only if it is "realizable" by a planar weighted directed acyclic network. In Section 3, we prove (Theorem 3.4) an analogous statement for loop groups: a matrix $X \in GL_n(\mathbb{R}[t, t^{-1}])$ is totally nonnegative if and only if it is "realizable" by a weighted directed acyclic network on a *cylinder* (see for example Fig. 4).

In the classical (planar) case, the minors of the matrix $X \in GL_n(\mathbb{R})$ are interpreted in terms of non-intersecting families of paths. Using the winding number of paths on a cylinder, we define a notion of pairs of paths being "uncrossed" (*not* the same as non-crossing). The analogous interpretation (Theorem 3.2) of minors of $X \in GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ involves uncrossed families of paths on the cylinder, and includes *some* paths which do intersect.

The idea of using a chord on a cylinder to keep track of the winding number, as it is done in this paper, appeared first in the work of Gekhtman et al. [15], which remained unpublished for some time.

1.8. The factorization problem

In [3], Berenstein et al. study the problem of finding an expression for the parameters $t_1, t_2, \ldots, t_\ell \in \mathbb{R}_{>0}$ in terms of the matrix entries of $X = e_{i_1}(t_1)e_{i_2}(t_2)\cdots e_{i_\ell}(t_\ell)$. They solve the problem by writing the parameters t_i as ratios of minors of the "twisted matrix" of X. This inverse problem led to the study of double wiring diagrams and double Bruhat cells [11], and later contributed to the discovery of cluster algebras [13].

In Section 10, we pose and solve a similar question in our setting. For a matrix X which is an infinite product of curls, we identify a particular factorization into curls, called the *ASW factorization*. Roughly speaking, the ASW factorization has curls ordered by radius of convergence. We express (Theorem 10.1 and Corollary 10.2) the parameters of the curls in the ASW factorization as limits of ratios of minors of X. Other factorizations of X into curls are obtained from the ASW factorization by the action of the infinite symmetric group S_∞.

1.9. Loop symmetric functions

One of the technical tools we use throughout the paper is a theory of tableaux for a Hopf algebra we call *loop symmetric functions*, denoted LSym. For $n = 1$, we obtain the usual symmetric functions. Roughly speaking, LSym generalizes usual symmetric functions in the same way matrix multiplication generalizes scalar multiplication. The points of $GL_n(\mathbb{R}((t)))_{\geq 0}$ are in bijection with algebra homomorphisms $\phi : \mathrm{LSym} \to \mathbb{R}$ which take nonnegative values on a particular spanning set of LSym. We leave the detailed investigation of LSym for future work, see [20] for a short survey. In the present article we define LSym analogues of homogeneous and elementary symmetric functions, tableaux, and Schur functions, and give a Jacobi–Trudi formula (Theorem 7.4).

1.10. Curl commutation relations, birational R-matrix, and discrete Painlevé systems

The commutation relations for curls give rise to a birational action of the symmetric group on a polynomial ring, for which LSym is the ring of invariants. This birational action was studied extensively by Noumi and Yamada [32,42] in the context of discrete Painlevé dynamical systems (see also [19]). It also occurs as a birational R-matrix in the Berenstein–Kazhdan [4] theory of geometric crystals (see also [10]). The tropicalization of this birational action is the combinatorial R-matrix of affine crystals, studied in [17].

We hope to clarify these unexpected connections in the future. The current progress in these directions is as follows. In [25,23] we construct certain affine geometric crystals in the unipotent loop group and use them to give a subtraction-free formula for energy function of some classical affine crystals. The curl commutation relation plays a crucial role. In [24] we study total positivity and geometric crystals on arbitrary orientable surfaces, with the cylinder corresponding to the current case of loop groups. We use our network techniques to generalize a result of Kajiwara et al. [16] on commuting R-matrix actions. The network model of [24] also leads to a generalization of discrete dynamical systems called ball-box systems. This generalization is studied in a joint work with Sakamoto [27]. Finally, a remarkable similarity between totally positive networks and electrical networks is explored in [26]. There we use an analog of curl commutation relation to solve the inverse Dirichlet-to-Neumann problem for certain cylindrical electrical networks.

1.11. Future directions

Our work suggests many future directions. For example:

What asymptotic representation theory corresponds to total nonnegativity of the formal loop group? (see [39,40,34,33]).
How does our work generalize to loop groups of other types? (see [31]).
Is there an "asymptotic" notion of a cluster algebra? (see [13]).

We also give a list of precise problems, conjectures and questions in Section 11.

2. The totally nonnegative part of the loop group

2.1. Formal and polynomial loop groups

An integer $n \geq 1$ is fixed throughout the paper. If $i \in \mathbb{Z}$, we write $\bar{i}$ for the image of i in $\mathbb{Z}/n\mathbb{Z}$. Occasionally, $\bar{i}$ is treated as an element of $\mathbb{Z}$, in which case we pick the representatives of $\mathbb{Z}/n\mathbb{Z}$ in $\{1, 2, \ldots, n\}$.

Let $GL_n(\mathbb{R}((t)))$ denote the *formal loop group*, consisting of $n \times n$ matrices $A(t) = (a_{ij}(t))_{i,j=1}^n$ whose entries are formal Laurent series of the form $a_{ij}(t) = \sum_{k \geq -N}^{\infty} b_k t^k$, for some real numbers $b_k \in \mathbb{R}$ and an integer N, and such that $\det(A(t)) \in \mathbb{R}((t))$ is a non-zero formal Laurent series. We let $GL_n(\mathbb{R}[t, t^{-1}]) \subset GL_n(\mathbb{R}((t)))$ denote the polynomial loop group, consisting of $n \times n$ matrices with Laurent polynomial coefficients, such that the determinant is a non-zero monomial. We will allow ourselves to think of the rows and columns of $A(t)$ to be labeled by $\mathbb{Z}/n\mathbb{Z}$, and if no confusion arises we may write $a_{ij}(t)$ for $a_{\bar{i}\bar{j}}(t)$, where $i, j \in \mathbb{Z}$.

To a matrix $A(t) = (a_{ij}(t)) \in GL_n(\mathbb{R}((t)))$, we associate a doubly-infinite, periodic, real matrix $X = (x_{i,j})_{i,j=-\infty}^{\infty}$ satisfying $x_{i+n,j+n} = x_{i,j}$ for any i, j, called the *unfolding* of $A(t)$, defined via the relation:

$$a_{ij}(t) = \sum_{k=-\infty}^{\infty} x_{i,j+kn} t^k.$$

We call $A(t)$ the *folding* of X, and write $A(t) = \overline{X}(t)$ for this relation. Clearly, $\overline{X}(t)$ and X determine each other and furthermore we have $XY = Z$ if and only if $\overline{X}(t)\overline{Y}(t) = \overline{Z}(t)$. We abuse notation by writing $X \in GL_n(\mathbb{R}((t)))$ or $X \in GL_n(\mathbb{R}[t, t^{-1}])$ if the same is true for $\overline{X}(t)$. If $X \in GL_n(\mathbb{R}((t)))$, we also write $\det(X)$ for $\det(\overline{X}(t))$. We define the *support* of X to be the set $\mathrm{supp}(X) = \{(i, j) \in \mathbb{Z}^2 \mid x_{ij} \neq 0\}$.

Example 2.1. For $n = 2$, an element of $GL_n(\mathbb{R}((t)))$ and its unfolding are

$$\begin{pmatrix} \cosh(\sqrt{abt}) & \sqrt{a/bt}\,\sinh(\sqrt{abt}) \\ \sqrt{bt/a}\,\sinh(\sqrt{abt}) & \cosh(\sqrt{abt}) \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & a & \frac{ab}{2} & \frac{a^2b}{6} & \frac{a^2b^2}{24} & \cdots \\ \cdots & 0 & 1 & b & \frac{ab}{2} & \frac{ab^2}{6} & \cdots \\ \cdots & 0 & 0 & 1 & a & \frac{ab}{2} & \cdots \\ \cdots & 0 & 0 & 0 & 1 & b & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Example 2.2. For $n = 3$, an element of $GL_n(\mathbb{R}[t, t^{-1}])$ and its unfolding are

$$\begin{pmatrix} 3 & 1 & 2t^{-1} \\ 1+t & 2 & 1 \\ t & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 2 & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 2 & 3 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 2 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For a real parameter $a \in \mathbb{R}$ and an integer k, we define $e_k(a) = (x_{i,j})_{i,j=-\infty}^{\infty} \in GL_n(\mathbb{R}[t, t^{-1}])$ to be the matrix given by

$$x_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ a & \text{if } j = i+1 \text{ and } \bar{i} = \bar{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, define $f_k(a) \in GL_n(\mathbb{R}[t, t^{-1}])$ to be the transpose of $e_k(a)$. We call the e_k-s and f_k-s *Chevalley generators*.

2.2. Totally nonnegative matrices

If $X \in GL_n(\mathbb{R}((t)))$, and $I \subset \mathbb{Z}$ and $J \subset \mathbb{Z}$ are finite sets of equal cardinality, we write $\Delta_{I,J}(X)$ for the minor of X obtained from the rows indexed by I and columns indexed by J. We write $X_{I,J}$ to denote a submatrix, so that $\det(X_{I,J}) = \Delta_{I,J}(X)$.

Let us say that $X \in GL_n(\mathbb{R}((t)))$ is *totally nonnegative*, or TNN for short, if every finite minor of X is nonnegative. We write $GL_n(\mathbb{R}((t)))_{\geq 0}$ for the set of totally nonnegative elements of $GL_n(\mathbb{R}((t)))$. Similarly, we define $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$. We say that $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is *totally positive* if there exists an integer k such that for every pair of subsets $I = \{i_1 < i_2 < \cdots < i_r\} \subset \mathbb{Z}$ and $J = \{j_1 < j_2 < \cdots < j_r\} \subset \mathbb{Z}$ satisfying $i_t \leq j_t + k$ for each $t \in [1, r]$, we have $\Delta_{I,J}(X) > 0$. In other words, X is totally positive if every sufficiently northeast minor is strictly positive. We denote the totally positive part of $GL_n(\mathbb{R}((t)))$ by $GL_n(\mathbb{R}((t)))_{>0}$. Note that $GL_n(\mathbb{R}((t)))_{>0} \cap GL_n(\mathbb{R}[t, t^{-1}]) = \emptyset$.

Example 2.3. The matrices in both Examples 2.1 and 2.2 are totally nonnegative. The matrix in Example 2.1 can be shown to be totally positive.

Lemma 2.1. *The sets $GL_n(\mathbb{R}((t)))_{\geq 0}$, $GL_n(\mathbb{R}((t)))_{>0}$ and $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ are semigroups.*

Proof. Follows immediately from the Cauchy–Binet formula which states that

$$\Delta_{I,J}(XY) = \sum_K \Delta_{I,K}(X)\Delta_{K,J}(Y) \tag{1}$$

where the sum is over subsets $K \subset \mathbb{Z}$ with the same cardinality as I and J.

Lemma 2.2. *Suppose $X \in GL_n(\mathbb{R}((t)))$. Then the rows of X, considered as vectors in $\mathbb{R}^\infty$, are linearly independent.*

Proof. Assume the statement is false and $\sum_{i \in I} p_i \mathbf{r}_i = 0$, where I is a finite set of rows, $p_i \in \mathbb{R}$ are real coefficients, and $\mathbf{r}_i$ denotes the i-th row of X. Then the rows $\overline{r}_j$ of the folding $\overline{X}$ satisfy $\sum_{i \in I} p_i t^{i'} \overline{r}_{\bar{i}} = 0$, where i' is defined by $i - i'n \in \{1, 2, \ldots, n\}$. But this implies that the rows of $\overline{X}$ are linearly dependent over $\mathbb{R}((t))$, contradicting the assumption that $\det(\overline{X})$ is non-vanishing.

A *solid minor* of a matrix is a minor consisting of consecutive rows and columns. A *row-solid minor* (resp. *column-solid minor*) is a minor consisting of consecutive rows (resp. consecutive columns).

Lemma 2.3. *Suppose $X \in GL_n(\mathbb{R}((t)))$. Then X is TNN if either all row-solid minors of X, or all column-solid minors of X, are nonnegative.*

Proof. Let M be a rectangular matrix with at least as many columns as rows. By a theorem of Cryer [7], such a matrix M of maximal rank is totally nonnegative if all its row-solid minors are totally nonnegative, cf. [2, Theorem 2.1]. By Lemma 2.2 we know that every minor of X is contained in a finite matrix of maximal rank formed by several consecutive rows of X, and we may assume that this finite matrix has more columns than rows. Thus to conclude nonnegativity of this minor it suffices to know nonnegativity of the row-solid minors of X. The same argument proves the statement for column-solid minors.

Throughout this paper, we will use the following naive topology on $GL_n(\mathbb{R}((t)))$. Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of infinite periodic matrices in $GL_n(\mathbb{R}((t)))$. Then $\lim_{k\to\infty} X^{(k)} = X$ if and only if $\lim_{k\to\infty} x_{ij}^{(k)} = x_{ij}$ for every i, j. We will show later in Proposition 4.4 that this seemingly weak notion of convergence implies much stronger notions for convergence in the case of TNN matrices.

Lemma 2.4. *Suppose X is the limit of a sequence $X^{(1)}, X^{(2)}, \ldots$ of TNN matrices. Then X is TNN.*

Proof. We must prove that every finite minor $\Delta_{I,J}(X)$ of X is nonnegative. But each such minor involves only finitely many entries. Thus $\Delta_{I,J}(X) = \lim_{i\to\infty} \Delta_{I,J}(X^{(i)}) \geq 0$.

For $X, Y \in GL_n(\mathbb{R}((t)))$, we write $X \leq Y$, if the same inequality holds for every entry. We note the following statement, which is used repeatedly.

Lemma 2.5. *Suppose X, Y and Z are nonnegative, upper-triangular matrices with 1's on the diagonal. Then $XYZ \geq XZ$.*

2.3. Semigroup generators for $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$

Let $T \subset GL_n(\mathbb{R}) \subset GL_n(\mathbb{R}((t)))$ denote the subgroup of diagonal matrices with real entries. Let $T_{>0}$ denote those diagonal matrices with positive real entries. Let $S = (s_{ij})_{i,j=-\infty}^{\infty} \in GL_n(\mathbb{R}((t)))$ denote the *shift matrix*, defined by

$$s_{ij} = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

The following is the loop group analogue of the Loewner–Whitney theorem (Theorem 1.1).

Theorem 2.6. *The semigroup $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ is generated by shift matrices, the positive torus $T_{>0}$ and Chevalley generators with positive parameters*

$$\{e_1(a), e_2(a), \ldots, e_n(a) \mid a > 0\} \cup \{f_1(a), f_2(a), \ldots, f_n(a) \mid a > 0\}.$$

Proof. First, using a (possibly negative) power of the shift matrix we can reduce to the case when the determinant of an element of $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ is a non-zero real number. Next, we recall (see [2]) that if

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a block decomposition of a finite square matrix M such that D is invertible, then the *Schur complement* $S(M, D)$ of the block D is the matrix $A - BD^{-1}C$ which has dimensions equal to that of A.

It is clear that all the generators stated in the Theorem do lie in $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$. Now let $X \in GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$. Call a non-zero entry $x_{i,j}$ of X a *NE corner (northeast corner)* if

$x_{i,j+k} = x_{i-k,j} = 0$ for $k \geq 1$. If $x_{i,j}$ is a NE corner then it follows from the TNN condition for size two minors that all entries strictly NE of $x_{i,j}$ all vanish.

A NE corner $x_{i,j}$ is *special* if $x_{i+1,j+1}$ is not a NE corner. We claim that either $x_{i,j} = 0$ for all $j > i$, or there exists a special NE corner. Indeed, if it was not so, that is if all NE corners lie along a diagonal $i - j = c > 0$ for some fixed c, then entries on this diagonal would contribute to $\det(\overline{X}(t))$ a monomial with a positive power of t not achieved by any other term in $\det(\overline{X}(t))$, leading to a contradiction.

Let $x_{i,j}$ be a special NE corner, which we may pick to be on a diagonal as NE as possible. We claim that $x_{i+1,j} > 0$. Indeed, if $x_{i+1,j} = 0$ then by nonnegativity of all 2×2 minors in rows $i, i+1$ we conclude that all entries in row $i+1$ of X are zero, contradicting the assumption that $X \in GL_n(\mathbb{R}[t, t^{-1}])$.

Now, let $X' = e_i\left(-\frac{x_{i,j}}{x_{i+1,j}}\right)X$. We claim that X' is again TNN (and it is clear that $X' \in GL_n(\mathbb{R}[t, t^{-1}])$). By Lemma 2.3 it suffices to check nonnegativity of row-solid minors, and in fact one only needs to check the row-solid minors containing row i of X but not the row $i+1$. Assume we have a row-solid minor with rows $I = [i', i]$ and column set J'. We may assume that $\max(J') \leq j$, for otherwise this minor will be 0 in both X and X'. Now pick a set of columns $J = [j', j]$ containing J'. Let Y be the rectangular submatrix of X with row set $[i', i+1]$ and columns set $[j', j]$. Complete it to a square matrix Z by adding zero rows or columns on the top or on the left. By construction Z is TNN and contains the row-solid minor we are interested in. Suppose that Z is a $m \times m$ matrix. Let Z' be obtained from Z by subtracting $\frac{x_{i,j}}{x_{i+1,j}}$ times the last row (indexed by $i+1$) from the second last row (indexed by i). Then the top left $(m-1)\times(m-1)$ submatrix of Z' is by definition equal to the Schur complement of $x_{i+1,j}$ in Z. It follows from [2, Theorem 3.3] that Z' is also TNN, and thus the minor of X' we are interested in has nonnegative determinant.

Note that the part of the support of X' above the main diagonal is strictly contained in that of X. On the other hand, the support below the main diagonal has not increased, as can be seen by looking again of positivity of 2×2 minors in rows $i, i+1$. Since after quotienting out by the periodicity the set $\mathrm{supp}(X)$ is finite, this process, when repeated, must terminate. That is, at some point we have $x_{i,j} = 0$ for all $j > i$. A similar argument with SW corners, and multiplication by f_j-s reduces X to a TNN matrix with entries only along the main diagonal. What remains is an element of $T_{>0}$, proving the theorem.

Example 2.4. The matrix in Example 2.2 factors as $f_3(2) f_1(1) e_2(1) e_1(1) e_3(1)$.

3. Cylindric networks and total positivity

3.1. Cylindric networks

Let $\mathfrak{C}$ be a cylinder (that is, $S^1 \times [0,1]$) and consider an oriented weighted network $N = (G, w, \mathfrak{h})$ on it defined as follows. G is a finite acyclic oriented graph embedded into $\mathfrak{C}$, having n sources $\{v_i\}_{i=1}^n$ on one of the two boundary components of $\mathfrak{C}$, and having n sinks $\{w_i\}_{i=1}^n$ on the other boundary component. Sources and sinks are numbered in counterclockwise order (we visualize the cylinder drawn standing with sources on the bottom and sinks on the top; "counterclockwise" is when viewed from above). We may, as usual, think of the sources and sinks as labeled by $\{v_i, w_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ and write v_i when we mean $v_{\bar{i}}$. The chord $\mathfrak{h}$ is a single edge connecting the two boundary components, starting on the arc $v_n v_1$ and ending on the arc $w_n w_1$. We assume $\mathfrak{h}$ is chosen so that no vertex of G lies on it.

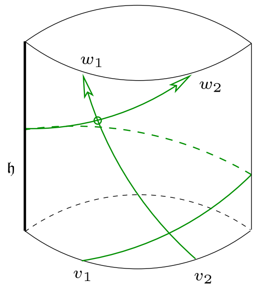

Fig. 1. A network on a cylinder.

The weight function $w : E(G) \longrightarrow \mathbb{R}_+$ assigns to every edge e of G a real nonnegative weight $w(e)$. The weight $w(p)$ of a path p is the product $\prod_{e \in p} w(e)$ of weights of all edges along the path. For a collection $P = \{p\}$ of paths we let $w(P) = \prod_{p \in P} w(p)$. For a path p let the *rotor* of p, denoted $\mathrm{rot}(p)$, be the number of times p crosses $\mathfrak{h}$ in the counterclockwise direction minus the number of times p crosses $\mathfrak{h}$ in the clockwise direction. If x, y are two vertices on a path p, we let $p_{[x,y]}$ denote the part of the path p between the points x and y, and let $*$ denote either the beginning or the end of a path. For example, $p_{[x,*]}$ denotes the part of p from x to the end of p.

For an integer i, let us define $\alpha(i) = (i - \bar{i})/n$, where $\bar{i}$ is to be taken in $\{1, 2, \ldots, n\}$. For two integers i and j, an (i, j)*-path* is a path in G which

(1) starts at the source $v_{\bar{i}}$;
(2) ends at the sink $w_{\bar{j}}$;
(3) has rotor equal to $\alpha(j) - \alpha(i)$.

Define an infinite matrix $X(N) = (x_{i,j})_{i,j=-\infty}^{\infty}$ by setting $x_{i,j}$ to be the sum of weights over all (i, j)-paths in G. Note that by definition $X(N)$ is periodic: $x_{i,j} = x_{i+n,j+n}$ for any i, j.

Let p be an (i, j)-path and let q be an (i', j')-path. Assume c is a point of crossing of p and q. Let $\tilde{p}$ and $\tilde{q}$ be the two paths obtained by swapping p and q at c: that is following one of them until point c and the other afterward. Although $\tilde{p}$ starts at $v_{\bar{i}}$ and ends at $w_{\bar{j}'}$, it is not necessarily an (i, j')-path, since $\mathrm{rot}(\tilde{p})$ may not be equal to $\alpha(j') - \alpha(i)$.

Example 3.1. Let $m = n = 4$ and consider two paths shown in Fig. 1, one an $(1, 6)$-path and one an $(2, 1)$-path. Then if we swap the two paths at the marked point of crossing, we do not get a $(1, 1)$-path and a $(2, 6)$-path. Instead we get a $(1, 5)$-path and a $(2, 2)$-path.

Lemma 3.1. *Let c be a point of intersection of p and q. Then the path $\tilde{p}$ is a (i, j')-path if and only if $\tilde{q}$ is a (i', j)-path. This happens when* $\mathrm{rot}(p_{[c,*]}) - \mathrm{rot}(q_{[c,*]}) = \alpha(j) - \alpha(j')$.

In the case of Lemma 3.1, we say that c is a *proper crossing* of p and q. Two paths that do not have a proper crossing we call an *uncrossed* pair of paths. Thus, the crossing marked in Fig. 1 is not proper. This pair of paths is however not uncrossed since the other crossing, not marked on the figure, happens to be proper.

3.2. *Cylindric Lindström Lemma*

Let $I = i_1 < \cdots < i_K$ and $J = j_1 < \cdots < j_K$ be two sets of indexes of equal (finite) cardinality K. Let $\Phi(I, J)$ denote the set of all families $P = \{p_k\}_{k=1}^{K}$ of paths such that

(1) each p_k is an (i_k, j_k)-path;
(2) every pair of paths in P are uncrossed.

The following theorem is a cylindric analogue of Lindström's Lemma [28].

Theorem 3.2. *We have*

$$\Delta_{I,J}(X(N)) = \sum_{P \in \Phi(I,J)} w(P).$$

First we prove the following lemma.

Lemma 3.3. *If $i < i'$ and $j' < j$ then every (i, j)-path p properly crosses every (i', j')-path q.*

Proof. We make use of the following observation: assume p and q are two paths that do not cross each other but might have one or two common endpoints. Then $\mathrm{rot}(p) - \mathrm{rot}(q)$ can only take values -1, 0, or 1.

Indeed, cut $\mathfrak{C}$ along p, viewing the result as a rectangle with a pair of opposite vertical sides identified. Since q never crosses p, it follows that q remains strictly inside the rectangle. Chord $\mathfrak{h}$ is represented inside the rectangle by at least $\mathrm{rot}(p) + 1$ disjoint segments. We can ignore the segments which have a crossing with the same vertical side of a rectangle, since their intersections with q contribute 0 to $\mathrm{rot}(q)$. What remains are exactly $\mathrm{rot}(p) + 1$ segments, all but the first and the last of which connect the two vertical sides of the rectangle. Those $\mathrm{rot}(p) - 1$ segments must be crossed by any path inside the rectangle, in particular by q. The first and the last segments of p however may or may not be crossed, depending on relative position of endpoints of p and q. This implies the needed statement concerning $\mathrm{rot}(p) - \mathrm{rot}(q)$.

We first claim that p and q have at least one point of intersection. This follows easily from unfolding the cylinder repeatedly. Let $c_1, \dots, c_k$ be all the crossings of p and q arranged in order. Now, by the argument above each of the quantities $a_0 = \mathrm{rot}(p_{[*,c_1]}) - \mathrm{rot}(q_{[*,c_1]})$, $a_1 = \mathrm{rot}(p_{[c_1,c_2]}) - \mathrm{rot}(q_{[c_1,c_2]}), \dots, a_k = \mathrm{rot}(p_{[c_k,*]}) - \mathrm{rot}(q_{[c_k,*]})$ is equal to -1, 0 or 1. Since

$$\mathrm{rot}(p) - \mathrm{rot}(q) = \sum_{m=0}^{k} a_m = \alpha(j) - \alpha(i) - \alpha(j') + \alpha(i') \geq \alpha(j) - \alpha(j') \geq 0$$

there must be an index $l \in \{0, 1, \dots, k+1\}$ such that $\sum_{m=l}^{k} a_m = \alpha(j) - \alpha(j')$. If $l \notin \{0, k+1\}$ then c_l is a proper crossing by Lemma 3.1. If $l = k+1$ then $\alpha(j) = \alpha(j')$ and $\bar{j} > \bar{j}'$, so as a result $a_k \leq 0$. Similarly, if $l = 0$ then $\alpha(i') - \alpha(i) = 0$ and $\bar{i}' > \bar{i}$, so as a result $a_0 \leq 0$. In both cases there exists at least one other index $l' \in \{1, \dots, k\}$ such that $\sum_{m=l}^{k} a_m = \alpha(j) - \alpha(j')$. It is easy to see that the resulting $c_{l'}$ is a proper crossing.

Now we are ready to prove Theorem 3.2.

Proof. Let P be a collection of K paths each of which is an (i_k, j_l)-path for some k, l so that each element of I and J is used once. Pick the first proper crossing c of two paths $p, q \in P$ (if it exists), where we choose an order on vertices of G according to some height function. We assume that the height function is chosen so that along any path the vertices are encountered in order of increasing height. We can of course assume without loss of generality that no two vertices of G have the same height. Now swap p and q after c, obtaining two new paths $\tilde{p}$ and $\tilde{q}$. Let $\tilde{P}$ be the collection obtained from P by replacing p, q with $\tilde{p}, \tilde{q}$. We claim that in $\tilde{P}$, c is again the first proper crossing of any pair of paths.

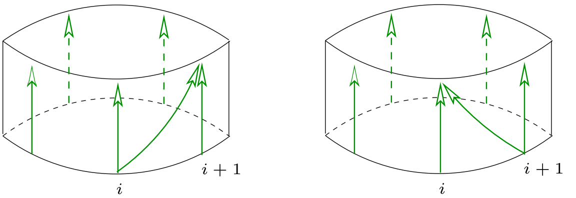

Fig. 2. Networks for Chevalley generators.

Assume p is an (i_k, j_l)-path and q is an $(i_{k'}, j_{l'})$-path. First, c is clearly a proper crossing of $\tilde{p}$ and $\tilde{q}$. We need to argue that it is still the first proper crossing. Suppose it is not. Since p and q are the only two paths in $\tilde{P}$ that changed, any possible new proper crossing $\tilde{c}$ preceding c must belong either to p or to q or to both.

If $\tilde{c}$ is a proper crossing of $\tilde{p}$ and $\tilde{q}$ then from $\mathrm{rot}(p_{[c,*]}) - \mathrm{rot}(q_{[c,*]}) = \alpha(j_l) - \alpha(j_{l'})$ and $\mathrm{rot}(\tilde{q}_{[\tilde{c},*]}) - \mathrm{rot}(\tilde{p}_{[\tilde{c},*]}) = \alpha(j_l) - \alpha(j_{l'})$ we obtain $\mathrm{rot}(p_{[\tilde{c},c]}) = \mathrm{rot}(q_{[\tilde{c},c]})$, from which it follows that $\tilde{c}$ should have been a proper crossing of p and q—this contradicts the original choice of c.

Similarly, suppose $\tilde{c}$ is a proper crossing of say $\tilde{q}$ and some r, which is an $(i_{k''}, j_{l''})$-path. Then $\mathrm{rot}(q_{[c,*]}) - \mathrm{rot}(p_{[c,*]}) = \alpha(j_l) - \alpha(j_{l'})$ and $\mathrm{rot}(\tilde{q}_{[\tilde{c},*]}) - \mathrm{rot}(r_{[\tilde{c},*]}) = \alpha(j_{l'}) - \alpha(j_{l''})$ imply $\mathrm{rot}(q_{[\tilde{c},*]}) - \mathrm{rot}(r_{[\tilde{c},*]}) = \alpha(j_l) - \alpha(j_{l''})$ and $\tilde{c}$ should have been a proper crossing of q and r.

Thus we have obtained a weight preserving involution on collections P of paths which have proper crossings. We observe looking at the corresponding terms of $\Delta_{I,J}(X(N))$ that this involution is sign-reversing. Thus, the corresponding contributions to the determinant cancel. To get the summation over $\Phi(I, J)$ it remains to check that a collection of paths is pairwise uncrossed only if each path in it is an (i_k, j_k)-path for some k. This follows from Lemma 3.3.

Remark 3.1. Theorem 3.2 and the other results in this section can be generalized to the case of n sources $\{v_i\}_{i=1}^n$ and m sinks $\{w_j\}_{j=1}^m$ in the obvious manner.

3.3. $GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$ *and cylindric networks*

Theorem 3.4. *Let $X \in GL_n(\mathbb{R}[t, t^{-1}])$. Then X is equal to $X(N)$ for some cylindric network N with nonnegative weight function, if and only if $X \in GL_n(\mathbb{R}[t, t^{-1}])_{\geq 0}$.*

Proof. From Theorem 3.2 it follows that every $X \in GL_n(\mathbb{R}[t, t^{-1}])$ that arises from a cylindric network is TNN. Further, concatenation of a cylindric network N and one of the special “building block” networks as shown in Figs. 2 and 3 corresponds to multiplication of $X(N)$ by a Chevalley generator and by a shift matrix respectively. We conclude by Theorem 2.6 that every element of $g_{\geq 0}$ can be represented by a cylindric network. ◻

3.4. *Determinant of the folding*

Let N be a cylindric network. We now give a combinatorial interpretation for the coefficients of the determinant $\det(\overline{X(N)}(t))$. Let $\{v_i\}_{i=1}^n$ and $\{w_i\}_{i=1}^n$ be the sources and sinks of N as before. Then $\overline{x}_{ij}(t)$ enumerates the weights of paths from v_i to w_j with an extra factor $t^{\mathrm{rot}(p)}$ keeping track of how many times the path p crossed the chord $\mathfrak{h}$ in the counterclockwise direction. Let Γ_k be the set of families $P = (p_1, \ldots, p_n)$ of paths, satisfying: (a) the path p_i connects v_i and $w_{\overline{i+k}}$, (b) no pair of paths intersect in the naive sense (rather than in the sense of “uncrossed” of Section 3.1), and (c) and there are k (net) counterclockwise crossings of paths in P with $\mathfrak{h}$.

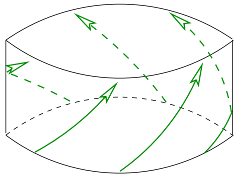

Fig. 3. A network for the shift matrix.

Theorem 3.5. *Let N be a cylindric network. Then*

$$\det(\overline{X(N)}(t)) = \sum_{k\in\mathbb{Z}} \left((-1)^{k(n-1)} \sum_{P\in\Gamma_k} w(P) \right) t^k.$$

Proof. We proceed using the usual argument in Lindström's lemma. Suppose $P = (p_1, p_2, \ldots, p_n)$ is a family of paths such that p_i goes from v_i to $w_{\sigma(i)}$ for some permutation $\sigma \in S_n$, and so that there are k (net) counterclockwise crossings of paths in P with $\mathfrak{h}$. If p_i and p_j intersect at a vertex c, swapping the two paths after c will give another family P' with the same weight, and still k (net) counterclockwise crossings with $\mathfrak{h}$. Applying the usual sign-reversing involution argument (see the proof of Theorem 3.2), we see that the coefficient of t^k in $\det(\overline{X(N)}(t))$ is equal to the weight generating functions of such families P with the additional requirement that no pair of paths intersect. We now observe such families P exist only if σ is a power of the long cycle, that is, belong to Γ_k. The sign of the corresponding permutation σ is $(-1)^{k(n-1)}$.

Example 3.2. Consider the network given in Fig. 4, where all edges are oriented upwards and have weight 1. One can check that the associated element of $GL_n(\mathbb{R}((t)))$ and its folding are given by

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 3 & 5 & 2 & 1 & 0 & \cdots \\ \cdots & 1 & 7 & 4 & 2 & 0 & \cdots \\ \cdots & 0 & 3 & 3 & 5 & 2 & \cdots \\ \cdots & 0 & 1 & 1 & 7 & 4 & \cdots \\ \cdots & 0 & 0 & 0 & 3 & 3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3+2t & 3t^{-1}+5+t \\ 1+4t & t^{-1}+7+2t \end{pmatrix}.$$

The determinant of the folded matrix equals $6 - t$. The non-crossing subnetwork corresponding to the $-t$ term is shown on the right of Fig. 4.

Corollary 3.6. *If $X = X(N)$ arises from a cylindric network N, then the odd minors of $\overline{X}(t)$ have nonnegative coefficients, the even minors have sign-alternating coefficients.*

4. Upper triangular matrices and a reduction result

4.1. Upper triangular matrices

Let $U \subset GL_n(\mathbb{R}((t)))$ be the subgroup of the formal loop group consisting of infinite periodic matrices which are upper triangular, and such that all diagonal entries are equal to 1. We denote the totally nonnegative matrices in U by $U_{\geq 0}$, and the totally positive matrices in U by $U_{>0}$.

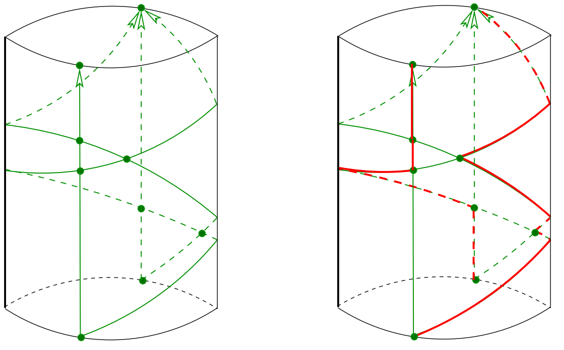

Fig. 4. A cylindric network and a non-crossing subnetwork.

We say that $X \in U_{\geq 0}$ is *finitely supported* if finitely many of diagonals of X, given by $j - i =$ constant, are non-zero. Otherwise we say that X is not finitely supported.

Lemma 4.1. *If $X \in U_{\geq 0}$ is not finitely supported then all of its entries above the main diagonal are non-zero.*

Proof. Suppose some entry $x_{i,j} = 0$. By using the nonnegativity of the 2×2 minors involving $x_{i,j}$ and either $x_{i,i}$ or $x_{j,j}$ we deduce that $x_{i,k} = 0$ for $k > j$ and $x_{k,j} = 0$ for $k < i$. Thus all the entries northeast of $x_{i,j}$ are 0. Since the entries of X are periodic, we deduce that X is finitely supported.

Thus if X is totally nonnegative but not finitely supported, then the entries of the folding of X are all infinite power series.

4.2. *Reduction to $U_{\geq 0}$*

Theorem 4.2. *Every $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ has a unique factorization of the form $X = FS^kY$ where F is the product of an element in $T_{>0}$ and some $f_i(a)$-s, k is an integer, and $Y \in U_{\geq 0}$.*

Proof. We first prove existence. By the definition of $GL_n(\mathbb{R}((t)))$, the matrix X has at least one SW-corner, where SW-corner is defined in obvious analogy with the NE-corners used in the proof of Theorem 2.6. Arguing as in that proof, either (a) one can write $X = f_j(a)X'$ where $X' \in GL_n(\mathbb{R}((t)))$ and $a > 0$, or (b) the southwestmost non-zero of diagonal of X is completely filled with non-zero entries. If we are in Case (b), then we can use the shift matrix S to shift the southwest-most diagonal to the central diagonal, and then multiply by a matrix in $T_{>0}$ to obtain the desired matrix $Y \in U_{\geq 0}$. In Case (a), we repeatedly factor out Chevalley generators $f_j(a)$, which in particular does not change the determinant $\det(X)$. We must eventually encounter Case (b), for otherwise we will have reduced the support of X to so far in the northeast that the lowest degree monomial in $\det(X)$ cannot be obtained. This establishes existence.

We now prove uniqueness. We first note that $ST_{>0}S^{-1} \in T_{>0}$ and that $Sf_i(a)S^{-1} = f_{i-1}(a)$. Suppose we have $FS^kY = F'S^{k'}Y'$. Then one has $Y'' = S^{k''}F''$ where $Y'' \in U$, F'' is a product of $f_i(a)$-s with possibly negative parameters, and $k'' \in \mathbb{Z}$. But $\det(Y'') \in 1 + t\mathbb{R}[[t]]$ and $\det(F'') \in \mathbb{R}$, so we conclude that $k'' = 0$. But F'' is lower triangular, and Y'' is upper triangular, so $F'' = Y'$ is the identity matrix. This implies that $k = k'$, $F = F'$, and $Y = Y'$.

For the rest of this section, and most of the rest of the paper, we focus on the semigroup $U_{\geq 0}$.

4.3. Convergence in $U_{\geq 0}$

Let $a(t) = 1 + a_1 t + a_2 t^2 + \cdots$ be a formal power series with real coefficients. Then $a(t)$ is a *totally positive function* if $a(t) = \overline{X}(t)$ for some $X \in U_{\geq 0}$ with $n = 1$. Note that with this terminology, we do not make the usual distinction between totally nonnegative and totally positive. As we have mentioned, the Edrei–Thoma theorem (Theorem 1.2) classifies totally positive functions.

Proposition 4.3. *Suppose $X \in U_{\geq 0}$. Then the entries of $\overline{X}(t)$ are meromorphic functions holomorphic in a neighborhood of* 0*.*

Proof. Apply Theorem 1.2 to each entry of $\overline{X}(t)$. (See also the proof of Proposition 4.4.)

The *radius of convergence* of X, denoted $r(X)$, is the minimum of the radii of convergence of the entries of $\overline{X}(t)$. The following proposition shows that our weak notion of convergence automatically implies stronger convergence.

Proposition 4.4. *Suppose $X^{(1)}, X^{(2)}, \ldots$ is a sequence of matrices in $U_{\geq 0}$ with limit X. Then there is a neighborhood $V \subset \mathbb{C}$ of* 0*so that*

(1) *every matrix amongst $\overline{X}^{(i)}(t)$ and $\overline{X}(t)$ is holomorphic in V*
(2) *every matrix entry of $\overline{X}^{(i)}(t)$ approaches the corresponding entry of $\overline{X}(t)$ uniformly, considered as holomorphic functions on V.*

Proof. It is enough to prove the statement for the case $n = 1$, that is, for totally positive functions. If $a(t) = 1 + a_1 t + \cdots$ is a totally positive function, then looking at 2×2 minors we have $a_1 \geq a_2/a_1 \geq a_3/a_2 \geq \cdots$, whenever the ratios are defined. Thus if $a(t)$ is not a polynomial, the radius of convergence $r(a)$ of $a(t)$ is at least a_i/a_{i+1} and we have $r = \lim_{i\to\infty} a_i/a_{i+1}$.

Now suppose that $a^{(1)}(t), a^{(2)}(t), \ldots$ converge to $a(t)$. Then there is a sufficiently large N so that for $k > N$, $|a_1^{(k)} - a_1| \leq 1$. It follows that $r(a^{(k)}(t)) > 1/(a_1 + 1)$ for all $k > N$ and so there exists a neighborhood V of 0 with property (1).

To see that $a^{(i)}(t)$ approaches $a(t)$ uniformly in a possibly smaller neighborhood V, we note that for $|t| < R$ we have

$$\left|\sum_{i\geq k} a_i t^i\right| \leq a_k R^k \sum_{i\geq k} a_1^{i-k} R^{i-k} \leq \frac{(a_1 R)^k}{1 - a_1 R}.$$

Fix some $R \ll 1/a_1$. It follows that for any $\ell \gg 0$, the value of $|a(t) - a^{(\ell)}(t)|$ for $|t| < R$ can be approximated by throwing away all but the first k terms. But for ℓ sufficiently large, the first k terms of $a(t)$ and $a^{(\ell)}(t)$ are arbitrarily close. This shows that $a^{(i)}(t)$ approaches $a(t)$ uniformly in the domain $|t| < R$.

Note that neither conclusion of Proposition 4.4 holds for general meromorphic functions.

4.4. The operation $^{-c}$

We define $X^c \in U$ to be the matrix obtained by applying to $X \in U$ the transformation $x_{i,j} \mapsto (-1)^{|i-j|} x_{i,j}$. A special role in what follows is played by the operation ^{c}*-inverse* given by $X \mapsto (X^c)^{-1}$. Abusing notation slightly, we shall also write $X^{-c} := (X^c)^{-1}$. Note that $(X^c)^{-1} = (X^{-1})^c$. Also note that the operation $X \mapsto X^{-c}$ is an involution, and that $(XY)^{-c} = Y^{-c} X^{-c}$.

Lemma 4.5. *Suppose $X \in U_{\geq 0}$. Then $X^{-c} \in U_{\geq 0}$.*

Proof. It suffices to show that $X^{-c}_{I,I}$ is TNN for every interval $I = [a,b]$, since every minor of X^{-c} is contained in such a submatrix. Let $Y = X_{I,I}$ and $m = |I|$. Then $Y \in GL_m(\mathbb{R})_{\geq 0} \subset GL_m(\mathbb{R}[t,t^{-1}])_{\geq 0}$. By Theorem 2.6 (or Theorem 1.1), Y is a product of Chevalley generators $\{e_i(a) \mid i = 1,2,\ldots,m-1\}$ with positive parameters. We now observe that $e_i(a)^{-c} = e_i(a)$. Using $(WV)^{-c} = W^{-c}V^{-c}$, we deduce that Y^{-c} is also a product of Chevalley generators with positive parameters. But then $X^{-c}_{I,I} = Y^{-c}$ is TNN.

Suppose i, j, k are integers such that $j - i - k \geq -1$ and $k \geq 0$. Let $X_{i,j,k}$ denote the solid submatrix of X obtained from the rows $i, i+1, \ldots, j-k$ and the columns $i+k, i+k+1, \ldots, j$. In the following proposition we use the convention that the determinant of a 0×0 matrix is 1.

Proposition 4.6. *Let $X \in U$. Then $\det(X_{i,j,k}) = \det(X^{-c}_{i,j,j+1-i-k})$ for $j \geq i+k-1$.*

Proof. If $k = 0$, then $\det(X_{i,j,k}) = 1 = \det(X^{-c}_{i,j,j+1-i-k})$. Consider now $k = 1$. That is, we need to show $(X^{-c})_{i,j} = \det(X_{i,j,1})$. Expanding $\det(X_{i,j,k})$ into smaller minors using the first row, we obtain

$$\det(X_{i,j,1}) = \sum_{r=0}^{j-i-1} (-1)^r x_{i,i+r+1} \det(X_{i+r+1,j,1}).$$

The claim then follows from the definition of X^{-c} and induction on $j - i$.

We now allow k to be arbitrary. We will prove the equality as a polynomial identity. Recall that for an $n \times n$ matrix M, Dodgson's condensation lemma [8] says

$$\begin{aligned} &\Delta(M)\Delta_{\{2,3,\ldots,n-1\},\{2,3,\ldots,n-1\}}(M) \\ &\quad = \Delta_{\{1,\ldots,n-1\},\{1,\ldots,n-1\}}(M)\Delta_{\{2,\ldots,n\},\{2,\ldots,n\}}(M) \\ &\qquad - \Delta_{\{2,\ldots,n\},\{1,\ldots,n-1\}}(M)\Delta_{\{1,\ldots,n-1\},\{2,\ldots,n\}}(M). \end{aligned} \tag{2}$$

Applying this and proceeding by induction on k, we calculate

$$\begin{aligned} &\det(X_{i,j,k+1}) \\ &\quad = \frac{\det(X_{i,j-1,k}) \cdot \det(X_{i+1,j,k}) - \det(X_{i,j,k}) \cdot \det(X_{i+1,j-1,k})}{\det(X_{i+1,j-1,k-1})} \\ &\quad = \frac{\det(X^{-c}_{i,j-1,j-i-k}) \cdot \det(X^{-c}_{i+1,j,j-i-k}) - \det(X^{-c}_{i,j,j+1-i-k}) \cdot \det(X^{-c}_{i+1,j-1,j-1-i-k})}{\det(X^{-c}_{i+1,j-1,j-i-k})} \\ &\quad = \det(X^{-c}_{i,j,j-i-k}). \end{aligned}$$

Note that the equalities hold as polynomials when applied to a matrix X consisting of variables $x_{i,j}$. Thus the divisions in the calculation are always legitimate.

Lemma 4.7. *We have*

$$\det(\overline{X^c})(t) = \begin{cases} \det(\overline{X})(t) & \text{if } n \text{ is even,} \\ \det(\overline{X})(-t) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Suppose $\overline{X}(t) = (\overline{x}_{ij}(t))$. Then $\overline{X^c}(t)$ has entries $(-1)^{j-i}\overline{x}_{ij}((-1)^n t)$.

5. Whirls, curls, and ASW factorization

5.1. Whirls and curls

Let $a_1, \ldots, a_n$ be n real parameters. We define a *whirl* to be a matrix $M = (m_{i,j})_{i,j=-\infty}^{\infty} = M(a_1, \ldots, a_n)$ with $m_{i,i} = 1, m_{i,i+1} = a_i$ and the rest of the entries equal to zero. Here, the indexing of the parameters are taken modulo n. Note that the Chevalley generator $e_i(a)$ is given by $M(0, \ldots, 0, a, 0, \ldots, 0)$ where the a is in the i-th position. If at least one of the parameters a_i in a whirl is zero, then we call the whirl *degenerate*. A degenerate whirl always factors into Chevalley generators. Furthermore, if the original parameters are nonnegative then the parameters in factorization are also nonnegative. We define a *curl* to be a matrix N of the form $N(a_1, \ldots, a_n) := M(a_1, \ldots, a_n)^{-c}$. Examples of whirls and curls were given in Section 1.

Lemma 5.1. *The folded determinants of whirls and curls are given by*

$$\det(M(a_1, \ldots, a_n)) = 1 + (-1)^{n+1} \left(\prod_{i=1}^{n} a_i \right) t$$

$$\det(N(a_1, \ldots, a_n)) = \frac{1}{1 - \left(\prod_{i=1}^{n} a_i \right) t}.$$

5.2. ϵ-sequence

Let $X \in U_{\geq 0}$. Define

$$\epsilon_i = \epsilon_i(X) = \lim_{j \longrightarrow \infty} \frac{x_{i,j}}{x_{i+1,j}}.$$

Clearly ϵ_i depends only on $\bar{i}$. Similarly, define

$$\mu_i = \mu_i(X) = \lim_{j \longrightarrow -\infty} \frac{x_{j,i+1}}{x_{j,i}}.$$

Example 5.1. Let $n = 2$. Consider the following matrix.

$$\begin{pmatrix} \dfrac{1+2t}{(1-t)(1-2t)} & \dfrac{2}{(1-t)(1-2t)} \\ \dfrac{3t}{(1-t)(1-2t)} & \dfrac{1+t}{(1-t)(1-2t)} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 2 & 5 & 6 & 13 & \cdots \\ \cdots & 0 & 1 & 3 & 4 & 10 & \cdots \\ \cdots & 0 & 0 & 1 & 2 & 5 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 3 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This matrix is in fact the product $N(1, 1)N(1, 2)$ of two curls, and thus is totally nonnegative. Then $\epsilon_1 = \lim_{i \to \infty} \frac{2^{i+2}-3}{3(2^i-1)} = \frac{4}{3}$. Similarly one computes $\epsilon_2 = \frac{3}{2}$.

Lemma 5.2. *Suppose $X \in U_{\geq 0}$ and not finitely supported. Then the limits ϵ_i and μ_i exist. Furthermore, $1/(\prod_{i=1}^n \epsilon_i) = 1/(\prod_{i=1}^n \mu_i)$ is the radius of convergence of every entry of the folding $\overline{X}(t)$.*

Proof. The inequality $\frac{x_{i,j}}{x_{i+1,j}} \geq \frac{x_{i,j+1}}{x_{i+1,j+1}}$ follows from the nonnegativity of the 2×2 minor $x_{i,j}x_{i+1,j+1} - x_{i+1,j}x_{i,j+1}$ of X. A non-increasing sequence of nonnegative real numbers has a limit, giving the first statement of the lemma. The second statement follows from the observation that

$$\frac{x_{i,j+n}}{x_{i,j}} = \frac{x_{i-n,j}}{x_{i,j}} = \prod_{k=0}^{n-1} \frac{x_{i+k-n,j}}{x_{i+k+1-n,j}}.$$

Although we often omit it from the notation, the ϵ_i-s depend on X. We call $(\epsilon_1, \ldots, \epsilon_n)$ the ϵ-sequence of X. Aissen et al. [1] used a factorization procedure as a first step toward the Edrei–Thoma theorem. We now describe a generalization of it to $n > 1$. We call this generalization *ASW factorization*.

Lemma 5.3. *Suppose $X \in U_{\geq 0}$ is not finitely supported. Let $X' = M(-\epsilon_1, \ldots, -\epsilon_n)X$. Then $X' \in U_{\geq 0}$.*

Proof. Let $J = j_1 < j_2 < \cdots < j_k$ be a set of column indices. We have

$$\lim_{l \longrightarrow \infty} \frac{\det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \cdots & x_{i,j_k} & x_{i,l} \\ x_{i+1,j_1} & x_{i+1,j_2} & \cdots & x_{i+1,j_k} & x_{i+1,l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k,j_1} & x_{i+k,j_2} & \cdots & x_{i+k,j_k} & x_{i+k,l} \\ x_{i+k+1,j_1} & x_{i+k+1,j_2} & \cdots & x_{i+k+1,j_k} & x_{i+k+1,l} \end{pmatrix}}{x_{i+k+1,l}}$$

$$= \lim_{l \longrightarrow \infty} \det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \cdots & x_{i,j_k} & x_{i,l}/x_{i+k+1,l} \\ x_{i+1,j_1} & x_{i+1,j_2} & \cdots & x_{i+1,j_k} & x_{i+1,l}/x_{i+k+1,l} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k,j_1} & x_{i+k,j_2} & \cdots & x_{i+k,j_k} & x_{i+k,l}/x_{i+k+1,l} \\ x_{i+k+1,j_1} & x_{i+k+1,j_2} & \cdots & x_{i+k+1,j_k} & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_{i,j_1} & x_{i,j_2} & \cdots & x_{i,j_2} & \epsilon_i \cdots \epsilon_{i+k} \\ x_{i+1,j_1} & x_{i+1,j_2} & \cdots & x_{i+1,j_2} & \epsilon_{i+1} \cdots \epsilon_{i+k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k,j_1} & x_{i+k,j_2} & \cdots & x_{i+k,j_2} & \epsilon_{i+k} \\ x_{i+k+1,j_1} & x_{i+k+1,j_2} & \cdots & x_{i+k+1,j_2} & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_{i,j_1} - \epsilon_i x_{i+1,j_1} & x_{i,j_2} - \epsilon_i x_{i+1,j_2} & \cdots & x_{i,j_k} - \epsilon_i x_{i+1,j_k} & 0 \\ x_{i+1,j_1} - \epsilon_{i+1} x_{i+2,j_1} & x_{i+1,j_2} - \epsilon_{i+1} x_{i+2,j_2} & \cdots & x_{i+1,j_k} - \epsilon_{i+1} x_{i+2,j_k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i+k,j_1} - \epsilon_{i+k} x_{i+k+1,j_1} & x_{i+k,j_2} - \epsilon_{i+k} x_{i+k+1,j_2} & \cdots & x_{i+k,j_k} - \epsilon_{i+k} x_{i+k+1,j_k} & 0 \\ x_{i+k+1,j_1} & x_{i+k+1,j_2} & \cdots & x_{i+k+1,j_k} & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} x_{i,j_1} - \epsilon_i x_{i+1,j_1} & x_{i,j_2} - \epsilon_i x_{i+1,j_2} & \cdots & x_{i,j_k} - \epsilon_i x_{i+1,j_k} \\ x_{i+1,j_1} - \epsilon_{i+1} x_{i+2,j_1} & x_{i+1,j_2} - \epsilon_{i+1} x_{i+2,j_2} & \cdots & x_{i+1,j_k} - \epsilon_{i+1} x_{i+2,j_k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{i+k,j_1} - \epsilon_{i+k} x_{i+k+1,j_1} & x_{i+k,j_2} - \epsilon_{i+k} x_{i+k+1,j_2} & \cdots & x_{i+k,j_k} - \epsilon_{i+k} x_{i+k+1,j_k} \end{pmatrix}.$$

This is a minor of X', and every row-solid minor of X' can be presented as a limit in this way. Since a limit of a nonnegative quantity is nonnegative, we conclude that all row-solid minors of X' are nonnegative. By Lemma 2.3, we conclude that X' is totally nonnegative.

We can rewrite the definition of X' as $X = N(\epsilon_1, \ldots, \epsilon_n)X'$. This gives a factorization of X into a product of two TNN matrices. Note that the radius of convergence of X' is at least as large as that of X. Thus, if we repeat the ASW factorization to obtain $X = N(\epsilon_1, \ldots, \epsilon_n)N(\epsilon'_1, \ldots, \epsilon'_n)X''$ then we must have $\prod_{i=1}^n \epsilon_i \geq \prod_{i=1}^n \epsilon'_i$. We also note that the factorization in Lemma 5.3 involves the "biggest" whirl.

Lemma 5.4. *Suppose $X \in U_{\geq 0}$ is not finitely supported. Suppose that*

$$X' = M(-a_1, \ldots, -a_n)X$$

is TNN. Then $a_i \leq \epsilon_i$ for each i. Furthermore, if $a_i < \epsilon_i$ for some i then $r(X') = r(X)$.

Proof. We obtain $X' = (x'_{i,j})$ from $X = (x_{i,j})$ by subtracting a_i times the $(i+1)$-th row from the i-th row. But the ratio $x_{i,j}/x_{i+1,j}$ approaches ϵ_i, so $x'_{i,j} \geq 0$ implies that $a_i \leq \epsilon_i$.

For the last statement, suppose that $a_i < \epsilon_i$. Since $r(M(-a_1, \ldots, -a_n)) = \infty$, we have $r(X') \geq r(X)$. But using Lemma 5.2, we have $r(N(a_1, \ldots, a_n)) = \prod_i \frac{1}{a_i} > \prod_i \frac{1}{\epsilon_i} = r(X)$ so that from $X = N(a_1, \ldots, a_n)X'$, we have $r(X) \geq r(X')$. Thus $r(X') = r(X)$.

Example 5.2. In Example 5.1, it was computed that the curl $N(\frac{4}{3}, \frac{3}{2})$ can be factored out on the left. One can check that the remaining totally nonnegative matrix is the curl $N(\frac{2}{3}, \frac{3}{2})$. Thus the ASW factorization of the matrix in this example is $N(\frac{4}{3}, \frac{3}{2})N(\frac{2}{3}, \frac{3}{2})$.

5.3. Finitely supported TNN matrices

Theorem 5.5. *The semigroup $U_{\geq 0}^{\mathrm{fin}}$ of finitely supported matrices in $U_{\geq 0}$ is generated by whirls and Chevalley generators with nonnegative parameters.*

Proof. It is clear that the semigroup generated by whirls and Chevalley generators with nonnegative parameters lies inside $U_{\geq 0}^{\mathrm{fin}}$. Now let $X \in U_{\geq 0}^{\mathrm{fin}}$. First suppose that X^{-c} is finitely supported. In this case, the entries of $\overline{X^{-c}}(t)$ are polynomials, and in particular, entire. But then both $1/\det(\overline{X^c}(t)) = \det(\overline{X^{-c}}(t))$ and $\det(\overline{X^c}(t))$ are polynomials, so we conclude that $\det(\overline{X^c}(t))$ and by Lemma 4.7 $\det(X)$ is a constant. By Theorem 2.6, we deduce that X fan be factored into a finite number of nonnegative Chevalley generators.

Now suppose that X^{-c} is not finitely supported. Apply Lemmas 4.5 and 5.3 to obtain $X^{-c} = N(a_1, \ldots, a_n)Y$, where the parameters $a_i = \epsilon_i(X^{-c})$ are nonnegative and Y is totally nonnegative. If at least one of parameters a_i is zero, by Lemma 5.2 the entries of $\overline{X^{-c}}$ are entire, and the determinant is entire. We may then proceed as in the case that X^{-c} is finitely supported.

Thus we may assume that all a_i are strictly positive. Then $X = Y^{-c}M(a_1, \ldots, a_n)$, where both X and Y^{-c} are finitely supported TNN matrices. One observes that the number of non-zero diagonals of Y^{-c} must be strictly smaller than that of X. Now repeat the application of Lemma 5.3 to Y^{-c}. Since the number of non-zero diagonals of X is finite, in a finite number of steps we must obtain the situation in one of the two previous paragraphs. Thus we obtain a factorization of X into a finite number of whirls and Chevalley generators with nonnegative parameters.

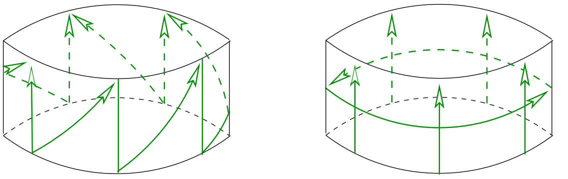

Fig. 5. Networks for a whirl and a curl.

Since whirls are representable by cylindric networks, as shown on the left in Fig. 5, we immediately get the following corollary.

Corollary 5.6. *Every $X \in U_{\geq 0}^{\mathrm{fin}}$ is representable by a cylindric network.*

5.4. Totally positive matrices

For $I = \{i_1 < i_2 < \cdots < i_k\}$ and $J = \{j_1 < j_2 < \cdots < j_k\}$ we define $I \leq J$ if $i_t \leq j_t$ for each $t \in [1, k]$.

Theorem 5.7. *Let $X \in U_{\geq 0}$. Then $X \notin U_{>0}$ if and only if X is a finite product of whirls and curls (including Chevalley generators). In other words, the semigroup generated by whirls and curls is exactly the set $U_{\geq 0} \setminus U_{>0}$.*

We start by proving the following lemma.

Lemma 5.8. *Suppose X has a vanishing minor $\Delta_{I,J}(X) = 0$ for $I \leq J$. Assume that (I, J) is chosen so that $|I| = |J| = k$ is minimal. Then X has a solid vanishing minor $\Delta_{I',J'}(X) = 0$ of size k with $I' \leq J'$. Furthermore, all minors $\Delta_{I'',J''}(X)$ for $I'' \leq I'$ and $J' \leq J''$ vanish.*

Proof. For $k = 1$ the statement is already proved in Lemma 4.1, so assume $k > 1$. If $I = i_1 < \cdots < i_k$ and $J = j_1 < \cdots < j_k$ then $i_k < j_k$ since otherwise there is a smaller singular minor. Look at the submatrix $X_{I\cup\{j_k\},J\cup\{j_k+1\}}$. Writing down Dodgson's condensation (2) for this matrix we get

$$-\Delta_{I\cup\{j_k\}-\{i_1\},J}(X)\Delta_{I,J\cup\{j_k+1\}-\{j_1\}}(X) = \Delta_{I\cup\{j_k\},J\cup\{j_k+1\}}(X)\Delta_{I-\{i_1\},J-\{j_1\}}(X).$$

This implies that the left-hand side must be zero, since it is non-positive and the right-hand side is nonnegative. If $\Delta_{I\cup j_k-\{i_1\},J}(X) = 0$ then the $(k-1)\times(k-1)$ minor $\Delta_{I-\{i_1\},J-\{j_k\}}(X)$ vanishes. If $I' = I - \{i_1\}$ and $J' = J - \{j_k\}$ satisfies $I' \leq J'$ then this contradicts the minimality of k. Otherwise we would have $i_{t+1} > j_t$ for some $t \in [1, k-1]$, implying that the submatrix $X_{I,J}$ is block upper triangular. Again this would imply a smaller vanishing minor, contradicting the minimality of k.

Thus $\Delta_{I,J\cup\{j_k+1\}-\{j_1\}}(X) = 0$. Repeating this k times, the column indexing set becomes solid, and similarly, we may move the rows up to obtain a solid row indexing set. The second claim is proved in a similar manner.

Corollary 5.9. *Suppose $X \in U_{\geq 0}$. Then $X \in U_{>0}$ if and only if all minors $\Delta_{I,J}(X) > 0$ for $I \leq J$.*

Lemma 5.10. *If $X \in U_{\geq 0}$ is not finitely supported and has a vanishing solid minor then all ϵ_i-s are positive.*

Proof. Let $k > 1$ be the size of smallest singular minor. It was shown in Lemma 5.8 that all, not necessarily solid, minors of size k far enough from the diagonal are singular. Consider the $k \times \infty$ submatrix $Y = X_{I,J}$ where $I = \{i, i+n, \ldots, i+(k-1)n\}$ and $J = \{j, j+n, \ldots\}$, where i and j are chosen so that all $k \times k$ minors of Y vanish. Thus Y has rank less than k. Since the $(k-1) \times (k-1)$ minors of Y do not vanish, there is a unique (up to scalar factor) linear relation between the rows of Y, say $\sum_{r=1}^{k} c_r \mathbf{y}_r = 0$, where y_r is the r-th row of Y, and all the c_r are non-zero.

We deduce that for large enough t we have $\sum_{r=1}^{k} c_r x_{i,j+(t-r+1)n}$. Then the limit $\delta = \lim_{t\to\infty} \frac{x_{i,j+tn}}{x_{i,j+(t-1)n}}$, which we know exists by Lemma 5.2, satisfies the polynomial equation $\sum_{r=1}^{k} c_r \delta^{k-r} = 0$. Since the c_r (in particular c_k) are all non-zero, $\delta \neq 0$. But δ is exactly the product of all ϵ_i-s (for $i = 1, 2, \ldots, n$).

Now we are ready to prove the theorem.

Proof of Theorem 5.7. Whirls, curls, and Chevalley generators all have the property that minors sufficiently far from the diagonal vanish. Thus any finite product of such matrices will have the same property. This shows that the semigroup generated by whirls and curls consists of totally nonnegative but not totally positive matrices.

Now suppose $X \in U_{\geq 0}$ is not totally positive. By Corollary 5.9, X has a vanishing minor $\Delta_{I,J}(X) = 0$ for $I \leq J$, which by Lemma 5.8 we may assume to be solid. We first suppose that (I, J) is chosen so that $I \leq J - 1$ and $\Delta_{I,J-1}(X) > 0$ (here $J - 1$ denotes $\{j_1 - 1, \ldots, j_k - 1\}$ where $J = \{j_1, \ldots, j_k\}$). This is possible because if I is not $\leq J - 1$, and both I, J are solid then $I = J$ and $\Delta_{I,J}(X)$ cannot vanish.

If X^{-c} is finitely supported, the statement follows from Theorem 5.5. If it is not finitely supported, we claim that ASW factorization (Lemma 5.3) factors a non-degenerate curl from X^{-c}. For that first note that if $I = (i+1, \ldots, i+k)$ and $J = (j+1, \ldots, j+k)$ then as was shown in the proof of Lemma 4.5 $\Delta_{I,J}(X) = \Delta_{I',J'}(X^{-c})$ where $I' = (i+1, \ldots, j)$ and $J' = (i+1+k, \ldots, j+k)$. Thus X^{-c} also has a singular solid minor with $I' \leq J'$. By Lemmatas 5.10 and 5.3, a non-degenerate curl N can be factored out from X^{-c}. We may thus write $X = X'M$ for a whirl $M = N^{-c}$ and totally nonnegative X'. We claim that in X' the minor $X'_{I,J-1}$ is singular. Indeed, in M the minor $M_{J-1,J}$ is non-singular. Then if $\Delta_{I,J-1}(X') > 0$ then by the Cauchy–Binet formula (1) we would have a positive term contributing to $\Delta_{I,J}(X)$, and since all other terms are nonnegative we obtain a contradiction.

Repeating this argument, the vanishing minor of X is moved closer and closer to the diagonal, so the process must eventually stop, at which point we will have obtained the desired factorization of X.

Note that curls can be represented by (non-acyclic) cylindric networks as shown on the right in Fig. 5. The definitions and results of Section 3 still hold when we allow oriented cycles with non-zero rotor in this way.

Corollary 5.11. *Every $X \in U_{\geq 0}$ which is not totally positive is representable by a finite cylindric network.*

5.5. Extension to the whole formal loop group

Proposition 5.12. *A matrix $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is totally positive if and only if the matrix $Y \in U_{\geq 0}$ of Theorem 4.2 is totally positive.*

Lemma 5.13. *Suppose $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ and $Y \in GL_n(\mathbb{R}((t)))_{>0}$. Then $XY, YX \in GL_n(\mathbb{R}((t)))_{>0}$.*

Proof. By Theorem 4.2, at least one of the diagonals of X has only non-zero entries. The statement follows easily.

Proof of Proposition 5.12. The "if" direction follows immediately from Lemma 5.13. For the other direction, it is enough to show that if $X \in U_{\geq 0}$ is not totally positive, and Y is a finitely supported matrix (such as FS^k in Theorem 4.2) then XY is not totally positive. By Lemma 5.8, all minors of X sufficiently far from the diagonal vanish. The statement then follows from the Cauchy–Binet formula (1).

Theorem 5.14. *A matrix $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is not totally positive if and only if it is a finite product of whirls, curls, upper or lower Chevalley generators, and shift matrices.*

Proof. The "only if" direction follows from Proposition 5.12 and Theorem 5.7. For, the "if" direction, all stated generators have all minors sufficiently northeast of the diagonal vanishing; that is all minors $\Delta_{I,J}$ where $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$ and $i_t \leq j_t - s$ for some s. Thus any finite product of such matrices will have the same property.

Example 5.3. We already know that the element of $GL_n(\mathbb{R}((t)))$ in Example 3.2 is representable by a cylindric network. We should also be able to factor it the way it is described in the theorem. Indeed, one can check that

$$f_1(1/3)f_2(9/16)T(9/8, 16/3)e_1(128/45)e_2(150/368)M(23/30, 5/23)$$

is one such factorization, where T denotes an element of the torus.

Corollary 5.15. *A matrix $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$ is not totally positive if and only if there exist s and k such that $\Delta_{I,J}(X) = 0$ whenever $I = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$ satisfy $i_t \leq j_t - s$.*

Corollary 5.16. *Suppose $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$. Then either every sufficiently large and sufficiently northeast minor of X vanishes, or every sufficiently northeast minor of X is positive.*

6. Whirl and curl relations

This section is concerned with relations that exist between products of whirls, curls and Chevalley generators. In the case $n = 1$ there are no Chevalley generators, while whirls and curls simply commute. For arbitrary n, we introduce a relation between products of two whirls or two curls, and another one between a whirl and a curl. We call these relations *commutation relations*, even though the factors do not commute. The commutation relations are well-defined only when one of the two factors is non-degenerate. If both factors are degenerate the commutation relations are not well-defined. However, in this case we may use the usual braid relations between Chevalley generators (see [31,21]).

Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{\geq 0}$ be two sets of parameters. Define

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k.$$

We call $\mathbf{a}$ degenerate if at least one of the a_i vanishes. Let $R \subset \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}$ be the subset of pairs $(\mathbf{a}, \mathbf{b})$ such that at most one of $\mathbf{a}$ and $\mathbf{b}$ is degenerate.

Now define a map $\eta : R \to R$ by $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ where

$$b_i' = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} \qquad a_i' = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

It is not hard to see that η is a well-defined map from R to R. For example, for $n = 3$ we have

$$b_1' = \frac{b_2(a_1a_3 + a_1b_3 + b_1b_3)}{a_2a_3 + b_2a_3 + b_2b_3}.$$

Lemma 6.1. *The function η has the following properties:*

(1) $a_i' + b_i' = a_i + b_i$;
(2) $b_i'a_{i+1}' = a_ib_{i+1}$;
(3) $\prod_{i=1}^n a_i = \prod_{i=1}^n a_i', \prod_{i=1}^n b_i = \prod_{i=1}^n b_i'$;
(4) *η is an involution.*

Proof. We have

$$\begin{aligned}
(a_i + b_i)\kappa_i(\mathbf{a}, \mathbf{b}) &= (a_i + b_i) \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k \\
&= a_i \sum_{j=i+1}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k + \prod_{i=1}^{n} a_i + b_i \\
&\quad \times \sum_{j=i}^{i+n-2} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k + \prod_{i=1}^{n} b_i \\
&= b_{i+1} \sum_{j=i+1}^{i+n-1} \prod_{k=i+2}^{j} b_k \prod_{k=j+1}^{i+n} a_k + \prod_{i=1}^{n} b_i + a_{i-1} \\
&\quad \times \sum_{j=i}^{i+n-2} \prod_{k=i}^{j} b_k \prod_{k=j+1}^{i+n-2} a_k \\
&\quad + \prod_{i=1}^{n} i = 1^n\, a_i = a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b}) + b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b}),
\end{aligned}$$

from which (1) follows. (2) and (3) are straight forward from the definition of η.

To prove (4), first suppose that $\mathbf{a}$ and $\mathbf{b}$ are both non-degenerate. Using (1) and (2), one can solve for b_1' to get

$$b_1' = a_1 + b_1 - \frac{a_nb_1}{a_n + b_n - \frac{a_{n-1}b_n}{\cdots - \frac{a_1b_2}{b_1'}}}.$$

This is a quadratic equation in b_1' and thus has at most two distinct solutions. Furthermore, it is clear that b_1', together with the values of $(a_i + b_i)$ and (a_ib_{i+1}) determine $(\mathbf{b}', \mathbf{a}')$ once (1) and (2) are known. So there are at most two solutions to (1) and (2) (with $(a_i + b_i)$ and (a_ib_{i+1}) fixed) which are $(\mathbf{a}, \mathbf{b})$ and $(\mathbf{b}', \mathbf{a}')$. Applying η to $(\mathbf{b}', \mathbf{a}')$ must again give one of these solutions. Now we observe that $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{a}, \mathbf{b})$ if and only if $\prod_i a_i = \prod_i b_i$. It thus follows from (3) that $\eta(\mathbf{b}', \mathbf{a}') = (\mathbf{a}, \mathbf{b})$.

Finally, the function $\eta^2(\mathbf{a}, \mathbf{b})$ is continuous, so the claim extends to the case that $\mathbf{a}$ or $\mathbf{b}$ is degenerate.

Theorem 6.2. *If* $\eta(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ *then* $M(\mathbf{a})M(\mathbf{b}) = M(\mathbf{b}')M(\mathbf{a}')$ *and* $N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')$.

Proof. The non-zero entries above diagonal in $M(a_1, \ldots, a_n)M(b_1, \ldots, b_n)$ are $a_i + b_i$ and $a_i b_{i+1}$. Now apply (1) and (2) from Lemma 6.1. The case of curls follows by taking $^{-c}$ of the whirl case.

Example 6.1. In Examples 5.1 and 5.2 we saw that $N(1,1)N(1,2) = N(\frac{4}{3}, \frac{3}{2})N(\frac{2}{3}, \frac{3}{2})$. Indeed, let us take $\mathbf{a} = (1,1)$ and $\mathbf{b} = (1,2)$. Then $\kappa_1(\mathbf{a}, \mathbf{b}) = 1 + 2 = 3$ and $\kappa_2(\mathbf{a}, \mathbf{b}) = 1 + 1 = 2$, which gives $b_1' = \frac{2\cdot 2}{3}$, $b_2' = \frac{1\cdot 3}{2}$, $a_1' = \frac{1\cdot 2}{3}$, $a_2' = \frac{1\cdot 3}{2}$ as desired.

If $(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)})$ is a sequence of n-tuples of nonnegative real numbers, we denote by $\eta_i(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)})$ the sequence of n-tuples obtained by applying η to $(\mathbf{a}^{(i)}, \mathbf{a}^{(i+1)})$ (assuming η is well-defined).

Theorem 6.3. *The map η satisfies the braid relation:*

$$(\eta_i \circ \eta_{i+1} \circ \eta_i)(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}) = (\eta_{i+1} \circ \eta_i \circ \eta_{i+1})(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)})$$

whenever the expressions are well-defined.

Proof. We may suppose $k = 3$, and consider a triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Since we are interested in the equality of two rational functions, it suffices to show that the statement is true for a Zariski dense set. We consider tuples $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ such that $\prod_i a_i > \prod_i b_i > \prod_i c_i$. Since this set locally looks like $\mathbb{R}^{3n}$, it is clear that it is Zariski dense. Let $(\mathbf{c}', \mathbf{b}', \mathbf{a}')$ and $(\mathbf{c}'', \mathbf{b}'', \mathbf{a}'')$ be the triples on the left and right hand side of the statement of the theorem.

Then using Lemma 6.1, we deduce $\prod_i a_i' = \prod_i a_i'' = \prod_i a_i$, $\prod_i b_i' = \prod_i b_i'' = \prod_i b_i$, $\prod_i c_i' = \prod_i c_i'' = \prod_i c_i$. Using Theorem 6.2, we have

$$X = N(\mathbf{c})N(\mathbf{b})N(\mathbf{a}) = N(\mathbf{a}')N(\mathbf{b}')N(\mathbf{c}') = N(\mathbf{a}'')N(\mathbf{b}'')N(\mathbf{c}'').$$

By assumption we have $r(X) = 1/(\prod_i a_i)$ (since $r(N(\mathbf{a})) = \prod_i 1/a_i$), and by Lemmas 5.2 and 5.4, we deduce that $\mathbf{a}' = \mathbf{a}''$. Similarly $\mathbf{b}' = \mathbf{b}''$ and $\mathbf{c}' = \mathbf{c}''$.

Corollary 6.4. *The $k-1$ maps $\eta_1, \eta_2, \ldots, \eta_{k-1}$ generate an action of S_k on $(\mathbb{R}^n_{>0})^k$.*

Proof. By Theorem 6.3 and Lemma 6.1, the maps satisfy the relations of the simple generators of the symmetric group S_k, and so generate an action of a subgroup of S_k. But if we pick a point $(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(k)}) \in (\mathbb{R}^n_{>0})^k$ such that $\prod_i a_i^{(1)} > \prod_i a_i^{(2)} > \cdots > \prod_i a_i^{(k)} > 0$ then the orbit of this point under the $k-1$ maps has size at least $k!$. Thus the maps generate an action of S_k.

Remark 6.1. Corollary 6.4 had previously been established in a number of different contexts: by Noumi and Yamada (see [32]) in the context of birational actions of affine Weyl groups, by Kirillov [19] in his study of tropical combinatorics, by Berenstein–Kazhdan [4] in the theory of geometrical crystals, and by Etingof [10] in the study of set-theoretical solutions of Yang–Baxter equations.

Now define $\theta : R \to R$ by $\theta(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ where

$$b_i' = \frac{(a_i + b_i)b_{i+1}}{a_{i+1} + b_{i+1}} \qquad a_i' = \frac{(a_i + b_i)a_{i+1}}{a_{i+1} + b_{i+1}}.$$

Lemma 6.5. *The function θ has the following properties:*

(1) $a_i + b_i = a'_i + b'_i$
(2) $\prod_i a_i = \prod_i a'_i$, $\prod_i b_i = \prod_i b'_i$
(3) *the inverse map* θ^{-1} *is given by* $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{b}', \mathbf{a}')$ *where*

$$b'_i = \frac{(a_i + b_i)b_{i-1}}{a_{i-1} + b_{i-1}} \qquad a'_i = \frac{(a_i + b_i)a_{i-1}}{a_{i-1} + b_{i-1}}$$

(4) *if* $\theta^{2k-1}(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ *then*

$$b'_i = \frac{(a_i + b_i)b_{i+2k-1}}{a_{i+2k-1} + b_{i+2k-1}} \qquad a'_i = \frac{(a_i + b_i)a_{i+2k-1}}{a_{i+2k-1} + b_{i+2k-1}}$$

(5) *if* $\theta^{2k}(\mathbf{a}, \mathbf{b}) = (\mathbf{a}', \mathbf{b}')$ *then*

$$a'_i = \frac{(a_i + b_i)a_{i+2k}}{a_{i+2k} + b_{i+2k}} \qquad b'_i = \frac{(a_i + b_i)b_{i+2k}}{a_{i+2k} + b_{i+2k}}$$

(6) $\theta^{\mathrm{lcm}(n,2)}$ *is the identity map.*

Proof. Statements (1)–(3) follow directly from definition, (4) and (5) are easily verified by induction, (6) follows from (4) and (5).

Theorem 6.6. *If* $\theta(\mathbf{a}, \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ *then* $M(\mathbf{a})N(\mathbf{b}) = N(\mathbf{b}')M(\mathbf{a}')$.

Proof. The (i, j)-th entry in $N(b'_1, \ldots, b'_n)M(a'_1, \ldots, a'_n)$ is

$$(b'_{j-1} + a'_{j-1}) \prod_{k=i}^{j-2} b'_k = (a_{j-1} + b_{j-1}) \prod_{k=i}^{j-2} \frac{(a_k + b_k)b_{k+1}}{a_{k+1} + b_{k+1}} = (a_i + b_i) \prod_{k=i+1}^{j-1} b_k$$

which is exactly the (i, j)-th entry of $M(a_1, \ldots, a_n)N(b_1, \ldots, b_n)$.

Both η and θ are well-defined as long as at least one of $\mathbf{a}$ and $\mathbf{b}$ is non-degenerate. The following lemma shows that interpreting a Chevalley generator as a degenerate whirl and using η results in the same relation as interpreting a Chevalley generator as a degenerate curl and using θ.

Lemma 6.7. *We have* $\eta((0, \ldots, 0, a_i, 0, \ldots, 0), \mathbf{b}) = \theta^{-1}((0, \ldots, 0, a_i, 0, \ldots, 0), \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ *and the map can be described as follows:*

(1) $b'_k = b_k, k \neq i+1, i$;
(2) $b'_{i+1} = \frac{b_{i+1}b_i}{a_i+b_i}$, $b'_i = a_i + b_i$;
(3) $a'_k = 0, k \neq i+1$;
(4) $a'_{i+1} = \frac{b_{i+1}a_i}{a_i+b_i}$.

Proof. The proof is a direct computation. For example, one has $\kappa_{i+2}(\mathbf{a}, \mathbf{b}) = \prod_{k \neq i+2} b_k$, $\kappa_{i+1}(\mathbf{a}, \mathbf{b}) = (a_i + b_i)\prod_{k \neq i,i+1} b_k$ and by definition $b'_{i+1} = b_{i+2}\kappa_{i+2}(\mathbf{a}, \mathbf{b})/\kappa_{i+1}(\mathbf{a}, \mathbf{b}) = b_{i+1}b_i/(a_i + b_i)$.

For later use we also give the following result.

Lemma 6.8. *The map* $\theta((0, \ldots, 0, a_i, 0, \ldots, 0), \mathbf{b}) = (\mathbf{b}', \mathbf{a}')$ *can be described as follows:*

(1) $b'_k = b_k, k \neq i-1, i$;
(2) $b'_{i-1} = \frac{b_{i-1}b_i}{a_i+b_i}$, $b'_i = a_i + b_i$;

(3) $a'_k = 0, k \neq i-1$;
(4) $a'_{i-1} = \frac{b_{i-1}a_i}{a_i+b_i}$.

Proof. The proof is a direct computation.

7. Infinite products of whirls and curls

7.1. Infinite whirls and curls

For a possibly infinite sequence of matrices $(X^{(i)})_{i=1}^{\infty}$ we write $\prod_{i=1}^{\infty} X^{(i)}$ for the limit

$$\lim_{k\to\infty} (X^{(1)}X^{(2)}\cdots X^{(k)}).$$

Similarly define $\prod_{i=-\infty}^{-1} X^{(i)}$ by

$$\lim_{k\to-\infty} (X^{(k)}X^{(k+1)}\cdots X^{(-1)}).$$

Lemma 7.1. *Let $(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}), (a_1^{(2)}, a_2^{(2)}, \ldots, a_n^{(2)}), \ldots$ be an infinite sequence of n-tuples of nonnegative numbers such that $\sum_{i=1}^{\infty}\sum_{j=1}^{n} a_j^{(i)} < \infty$. Then the limits*

$$\prod_{i=1}^{\infty} M(a_1^{(i)}, \ldots, a_n^{(i)}), \qquad \prod_{i=-\infty}^{-1} M(a_1^{(-i)}, \ldots, a_n^{(-i)}), \qquad \prod_{i=1}^{\infty} N(a_1^{(i)}, \ldots, a_n^{(i)}),$$
$$\prod_{i=-\infty}^{-1} N(a_1^{(-i)}, \ldots, a_n^{(-i)})$$

exist and are TNN matrices. Conversely, the limits exist only if the sum is finite.

Proof. We will prove the statement for $\prod_{i=1}^{\infty} M(a_1^{(i)}, \ldots, a_n^{(i)})$. The result for curls is obtained by taking inverses. Each entry of the sequence $\prod_{i=1}^{k} M(a_1^{(i)}, \ldots, a_n^{(i)})$ is non-decreasing as $k \to \infty$ so it suffices to prove that every entry is bounded. It is easy to see that the entries directly above the diagonal are bounded by $\alpha = \sum_{i=1}^{\infty}\sum_{j=1}^{n} a_j^{(i)}$. By induction, one sees that entries along the d-th diagonal are bounded by α^d.

By Lemma 5.2 we see that $\prod_{i=1}^{\infty} M(a_1^{(i)}, \ldots, a_n^{(i)})$ is TNN.

We call the products above *right-infinite whirls*, *left-infinite whirls*, *right-infinite curls* and *left-infinite curls*. If X is an infinite whirl (resp. curl) we say that X is of *whirl type* (resp. *curl type*).

Lemma 7.2. *Let X one of the infinite products in Lemma 7.1. Then the folded determinant of X is given by*

$$\det(\overline{X}(t)) = \begin{cases} \displaystyle\prod_{i=1}^{\infty}\left(1 + (-1)^{n+1}\left(\prod_{j=1}^{n} a_j^{(i)}\right) t\right) & \text{if } X \text{ is of whirl type} \\ \displaystyle\prod_{i=1}^{\infty}\frac{1}{1-\left(\prod_{j=1}^{n} a_j^{(i)}\right) t} & \text{if } X \text{ is of curl type.} \end{cases}$$

Proof. Each coefficient of $\det(\overline{X}(t))$ depends on only finitely many entries of X. The statement then follows from taking an infinite product of Lemma 5.1.

7.2. *Loop symmetric functions*

In this subsection, we assume familiarity with the theory of Young tableaux and symmetric functions [38]. Let $Y = (y_{k,l})_{k,l=-\infty}^{\infty} = \prod_{i=1}^{\infty} N(\mathbf{x}_i)$ be a right-infinite curl, where $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(n)})$. Note that in order to agree with usual symmetric function conventions, we have labeled (in this subsection only) the upper and lower indices of the curl parameters $x_i^{(j)}$ in the opposite manner to our usual notation. We caution the reader that with variables $a_j^{(i)}$ it is the lower index that is in $\mathbb{Z}/n\mathbb{Z}$.

We now interpret the entries of Y as analogs of homogeneous symmetric functions in variables $x_i^{(j)}$. Define for each $r \geq 1$ and each $k \in \mathbb{Z}/n\mathbb{Z}$,

$$h_r^{(k)}(\mathbf{x}) = \sum_I x_{i_1}^{(k)} x_{i_2}^{(k+1)} \cdots x_{i_r}^{(k+r-1)}$$

where the sum is taken over all weakly increasing sequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r$. We shall call the $h_r^{(k)}(\mathbf{x})$ *(curl) loop homogeneous symmetric functions*, to be distinguished from the whirl loop symmetric functions which will appear later.

Lemma 7.3. *Let $Y = \prod_{i=1}^{\infty} N(\mathbf{x}_i)$. Then we have $y_{k,l} = h_{l-k}^{(k)}(\mathbf{x})$.*

Proof. We first argue the statement is valid for any finite number of curls. We proceed by induction, the case of one curl follows trivially from the definition of curls. Assume we have already shown that the entries of $Y = \prod_{i=1}^{m-1} N(x_i^{(1)}, \ldots, x_i^{(n)})$ are described by the stated formula. Let us consider $Y' = YN$ where $N = N(x_m^{(1)}, \ldots, x_m^{(n)})$. We have $y'_{k,l} = \sum_{t=0}^{l-k} y_{k,k+t} N_{k+t,l}$. We know that $y_{k,k+t}$ equals the sum $\sum_I x_{i_1}^{(k)} x_{i_2}^{(k+1)} \cdots x_{i_t}^{(k+t-1)}$ over all weakly increasing sequences I of length t. At the same time $N_{k+t,l}$ equals the product $x_m^{(k+t)} x_m^{(k+t+1)} \cdots x_m^{(l)}$. Thus the term $y_{k,k+t} N_{k+t,l}$ of the summation equals the sum $\sum_{I'} x_{i_1}^{(k)} x_{i_2}^{(k+1)} \cdots x_{i_{l-k}}^{(l-1)}$ over all sequences

$$I' = i_1 \leq i_2 \leq \cdots \leq i_{k+t-1} < i_{k+t} = \cdots = i_l = m.$$

Summing over t gives the desired result.

For an infinite product of curls, the result follows from taking the limit $m \to \infty$. The limit exists by Lemma 7.1.

Now we provide an analog of Jacobi–Trudi formula, giving an interpretation for minors of Y as generalizations $s_\lambda(\mathbf{x})$ of skew Schur functions, which we call *(curl) loop Schur functions*. Let $\lambda = \rho/\nu$ be a skew shape, which we shall draw in the English notation:

A square $s = (i, j)$ in the i-th row and j-th column has *content* $j - i$ and has *(curl) residue* $r(s) = \overline{j - i} \in \mathbb{Z}/n\mathbb{Z}$. Recall that a semistandard Young tableaux T with shape λ is a filling of each square $s \in \lambda$ with an integer $T(s) \in \mathbb{Z}_{>0}$ so that the rows are weakly-increasing, and columns are increasing. An example of a semistandard tableau is given on the right in Fig. 6. The weight x^T of a tableaux T is given by $x^T = \prod_{s \in \lambda} x_{T(s)}^{(r(s))}$. We define the (curl) loop Schur function by

$$s_\lambda(\mathbf{x}) = \sum_T x^T$$

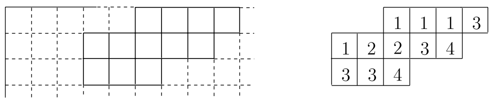

Fig. 6. A semistandard tableau.

where the summation is over all semistandard Young tableaux of (skew) shape λ. We shall also need several alternative definitions. We define the *whirl residue* $\overline{r}(s) = \overline{i-j} \in \mathbb{Z}/n\mathbb{Z}$. We define $\overline{x}^T = \prod_{s\in\lambda} x_{T(s)}^{(\overline{r}(s))}$ and the *whirl loop Schur functions*

$$\overline{s}_\lambda(\mathbf{x}) = \sum_T \overline{x}^T.$$

Theorem 7.4. *Let $Y = \prod_{i=1}^\infty N(\mathbf{x}_i)$. Let $I = i_1 < i_2 < \cdots < i_k$ and $J = j_1 < j_2 < \cdots < j_k$ be two sequences of integers such that $i_t \le j_t$. Define*

$$\lambda = \lambda(I, J) = (j_k, j_{k-1}+1, \ldots, j_1+k-1)/(i_k, i_{k-1}+1, \ldots, i_1+k-1).$$

Then

$$\Delta_{I,J}(Y) = \det\bigl(h_{j_t-i_s}^{(i_s)}(\mathbf{x})\bigr)_{s,t=1}^k = s_\lambda(\mathbf{x}).$$

Note that if I and J do not satisfy the condition $i_t \le j_t$ then $\Delta_{I,J}(Y) = 0$.

Proof. The first equality follows from Lemma 7.3. We prove the second inequality using the Gessel–Viennot method in the standard manner. We refer the reader to [38, Chapter 7] for details concerning this method.

Consider the square lattice grid in the plane, and orient all vertical edges north and all horizontal edges east. Assign to vertical edges weight 1. Assign to a horizontal edge of the grid connecting (p, q) with $(p+1, q)$ the weight $x_{q+1}^{(p)}$. Consider k sources with coordinates $(i_s, 0)$, $s = 1, \ldots, k$ and k sinks with coordinates (j_t, ∞), $t = 1, \ldots, k$. One checks directly that the weight generating function of paths from the source $(i_s, 0)$ to (j_t, ∞) is equal to $h_{j_t-i_s}^{(i_s)}$. By the Gessel–Viennot method, the determinant $\det(h_{j_t-i_s}^{(i_s)})_{s,t=1}^k$ is the weight generating function of non-intersecting families of paths from these k sources to the k-sinks. It is easy to see that such families are in bijection with semistandard tableaux T of shape λ, and that the weight of the path family corresponding to a tableau T is exactly x^T.

Example 7.1. Let $n = 3$. For $I = (1, 2, 5)$ and $J = (4, 7, 9)$ we get the skew shape shown in Fig. 6. The monomial corresponding to the shown semistandard filling is

$$x_1^{(1)} x_2^{(1)} x_3^{(1)} (x_1^{(2)})^2 (x_3^{(2)})^3 x_1^{(3)} x_2^{(3)} (x_4^{(3)})^2.$$

We now state similar theorems for right-infinite whirls, and the proofs are completely analogous. Let $Y = \prod_{i\ge1} M(x_i^{(1)}, \ldots, x_i^{(n)})$ be a right-infinite whirl. We define the *whirl loop elementary symmetric functions* $\overline{e}_r^{(k)}(\mathbf{x}) = \sum_I x_{i_1}^{(k)} x_{i_2}^{(k+1)} \cdots x_{i_r}^{(k+r-1)}$, where the sum is taken over all increasing sequences $i_1 < i_2 < \cdots < i_r$.

Lemma 7.5. *Let $Y = \prod_{i=1}^\infty M(\mathbf{x}_i)$. We have $y_{k,l} = \overline{e}_{l-k}^{(k)}(\mathbf{x})$.*

If λ is a skew shape, we let λ' denote the conjugate of λ, obtained reflecting λ in the main diagonal.

Theorem 7.6. *Let $Y = \prod_{i=1}^{\infty} M(\mathbf{x}_i)$. Let $I = i_1 < i_2 < \cdots < i_k$ and $J = j_1 < j_2 < \cdots < j_k$ be two sequences of integers such that $i_t \leq j_t$. Define*

$$\lambda = \lambda(I, J) = (j_k, j_{k-1} + 1, \ldots, j_1 + k - 1)/(i_k, i_{k-1} + 1, \ldots, i_1 + k - 1).$$

Then

$$\Delta_{I,J}(Y) = \det(\bar{e}^{(i_s)}_{j_t - i_s}(\mathbf{x}))_{s,t=1}^{k} = \bar{s}_{\lambda'}(\mathbf{x}).$$

Remark 7.1. If we consider the $x_i^{(j)}$ as variables, then $\{h_r^{(k)}\}$ are algebraically independent (and so are the $\{\bar{e}_r^{(k)}\}$). The commutative ring which the $\{h_r^{(k)}\}$ generate we call the *(curl) loop symmetric functions*, denoted LSym. (The ring generated by the $\{\bar{e}_r^{(k)}\}$, called the *whirl loop symmetric functions*, is distinct from LSym, considered as subrings of the ring of formal power series.) The ring LSym is a Hopf algebra which coincides with the usual ring of symmetric functions when $n = 1$. We shall study LSym in detail in future work.

Remark 7.2. Our loop homogeneous symmetric functions also appear in the context of Noumi–Yamada's study of discrete Painlevé dynamical systems, see [42].

Remark 7.3. The concept of *chess tableaux* in the work of Scott [37] seems to be related to the weight of the tableaux as defined here.

7.3. Basic properties of infinite whirls and curls

We say that a matrix $A = A(t)$ is entire if every entry of A is entire. We say $X \in U$ is entire if $A(X)$ is.

Lemma 7.7. *Let $X = \prod_{i=1}^{\infty} M(\mathbf{a}^{(i)})$ (resp. $X = \prod_{i=-\infty}^{-1} M(\mathbf{a}^{(-i)})$) be well-defined as in Lemma 7.1 and not finitely supported. Then $\mu_i(X) = 0$ (resp. $\epsilon_i(X) = 0$) for each i. In particular, X is entire.*

We remind the reader that with the $\mathbf{a}$ variables, the lower index is the one taking values in $\mathbb{Z}/n\mathbb{Z}$.

Proof. Let us consider $X = \prod_{i=1}^{\infty} M(\mathbf{a}^{(i)})$; the other case is similar. Using Lemma 7.5 and the definition of $\mu_i(X)$, we must show for each k that the ratio $\bar{e}_{s+1}^{(k+s)}(\mathbf{a})/\bar{e}_s^{(k+s)}(\mathbf{a})$ converges to 0 as $s \to \infty$. We know that $\bar{e}_s^{(k+s)}(\mathbf{a})$ is the generating function of semistandard tableau with shape a column of size s and initial residue $k + s$. Given such a column tableau T with size $s + 1$ we may produce a column tableau T' with size s by removing the letter in the last (lowest) box. A fixed column tableau T' with size s can be obtained in this way for each possible value of the last box. But for sufficiently large s, we have $\sum_{i \geq s}^{\infty} \sum_{j=1}^{n} a_j^{(i)} < \varepsilon$, for any given $\varepsilon > 0$. Thus for sufficiently large s, we have $\bar{e}_{s+1}^{(k+s)}(\mathbf{a})/\bar{e}_s^{(k+s)}(\mathbf{a}) < \varepsilon$, as required.

Lemma 7.8. *Let $X = \prod_{i=1}^{\infty} N(\mathbf{a}^{(i)})$ or $X = \prod_{i=-\infty}^{-1} N(\mathbf{a}^{(-i)})$ be well-defined as in Lemma 7.1. Define $b_i = \prod_{j=1}^{n} a_j^{(i)}$ and assume that $b_1 = \max_i b_i \neq 0$. Then $\epsilon_j(X) = a_j^{(1)}$.*

Proof. We consider the case $X = \prod_{i=1}^{\infty} N(\mathbf{a}^{(i)})$. Let $\alpha = a_j^{(1)}$. The sum of all $a_j^{(i)}$ converges, so certainly $b_i \to 0$ and the maximum $b = \max_i b_i$ exists. By Lemma 5.4, we have $\epsilon_j(X) \geq \alpha$. By Lemmatas 7.3 and 5.2, it suffices to check that $\lim_{s\to\infty} h_{s+1}^{(j)}(\mathbf{a})/h_s^{(j+1)}(\mathbf{a}) \leq \alpha$. Let S_{s+1} be the set of semistandard tableaux of shape a row of length $s+1$, shifted in the plane so that the initial box has residue j. Similarly let S_s be the set of semistandard tableaux of shape a row of length s, with initial box having residue $j+1$. If S is a set of tableaux, then we write $\mathrm{wt}(S) = \sum_{T\in S} a^T$. Thus $\mathrm{wt}(S_{s+1}) = h_{s+1}^{(j)}(\mathbf{a})$ and $\mathrm{wt}(S_s) = h_s^{(j+1)}(\mathbf{a})$, so it suffices to prove that for sufficiently large s we have $\mathrm{wt}(S_{s+1}) \leq (a_j^{(1)} + \varepsilon)\mathrm{wt}(S_s)$ for arbitrarily small ε. Given a tableau $T \in S_s$ we can obtain a tableau $T' \in S_{s+1}$ by adding the number 1 in front, and we have $a^{T'} = \alpha \cdot a^T$. Let $S_s' \subset S_s$ be the subset of tableaux which start with a number 2 or greater, and let $S_s^* = S_s - S_s'$. It is enough to show that for sufficiently large s we have $\mathrm{wt}(S_s') \leq \varepsilon \mathrm{wt}(S_s^*)$ for arbitrarily small ε. (For tableaux $T \in S_s^*$ only the number 1 can be added in front, and every $T' \in S_{s+1}$ is obtained by adding some number in front of some $T \in S_s$.)

Pick R so that

$$\sum_{i\geq R}\sum_{j=1}^{n} a_j^{(i)} < \min(a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)}).$$

This can be done since the sum $\sum_{i,j} a_j^{(i)}$ is finite. Let $W \subset S_s'$ denote the tableaux labeled with numbers from $\{2, 3, \ldots, R\}$, where we now declare that for $T \in W$, the tableau has a modified weight wt': the number R in a square with residue j has weight $a_j^{(1)}$. By the construction of R, we deduce that $\mathrm{wt}(S_s') \leq \mathrm{wt}'(W)$ using this modified weight.

Pick $s > nR^2/\varepsilon$. Given a tableau $T \in W$ there are at least s/R (consecutive) numbers all equal to some $r \in [2, R]$. We pick the smallest such r. We define a collection $\gamma(T) \subset S_s^*$ by removing the first $n, 2n, \ldots,$ of these numbers from T, and replacing them with 1's in the beginning of T. Thus $\gamma(T)$ consists of at least s/nR distinct tableaux. Furthermore, each tableau in $\gamma(T)$ has weight greater than the (modified) weight of T, and each tableau in S_s^* can occur this way in at most R ways. We conclude that

$$(s/nR)\mathrm{wt}(S_s') \leq (s/nR)\mathrm{wt}'(W) \leq R\ \mathrm{wt}(S_s^*)$$

so that $\mathrm{wt}(S_s') \leq \varepsilon \mathrm{wt}(S_s^*)$, as required.

Lemma 7.9. *Let $X = \prod_{i=1}^{\infty} N(\mathbf{a}^{(i)})$ or $X = \prod_{i=-\infty}^{-1} N(\mathbf{a}^{(-i)})$ be well-defined as in Lemma 7.1. Define $b_i = \prod_{j=1}^{n} a_j^{(i)}$ and assume that $\max_i b_i \neq 0$. Then $r(X) = 1/(\max_i b_i)$.*

Proof. We prove the statement for $X = \prod_{i=1}^{\infty} N(\mathbf{a}^{(i)})$. Using Theorem 6.2 possibly repeatedly, we may assume that $b = b_1$ is maximal. The result then follows from Lemmatas 5.2 and 7.8.

Remark 7.4. The assumption $\max_i b_i \neq 0$ in Lemmatas 7.8 and 7.9 can be removed (see [21]).

Corollary 7.10. *Suppose X is of curl type. Then the radius of convergence of $\det(\overline{X})$ is equal to $r(X)$.*

8. Canonical form

Let $RC \subset U_{\geq 0}$ denote the set of matrices of the form $Z = \prod_{i=1}^{\infty} N(a_1^{(i)}, \ldots, a_n^{(i)})$ where all the $a_j^{(i)}$ are strictly positive and the sum of the $a_j^{(i)}$ converges. In other words, RC is the set of right-infinite products of non-degenerate curls. Define $\overline{RC}$ to be the union of RC and the set of finite products of non-degenerate curls. Similarly we define LC and $\overline{LC}$ (left-infinite non-degenerate curls), RW and $\overline{RW}$ (right-infinite non-degenerate whirls), and LW and $\overline{LW}$ (left-infinite non-degenerate whirls).

8.1. Whirl and curl components

Let $X \in U_{\geq 0}$. A *curl factorization* of X is a factorization of the form $X = ZY$, where Y is entire and $Z \in \overline{RC}$.

We say that a (possibly finite) sequence $X = X^{(0)}, X^{(1)}, \ldots$ of TNN matrices is a *curl reduction* of X if

(1) $X^{(i)} = N(a_1^{(i)}, \ldots, a_n^{(i)})X^{(i+1)}$ for each i, where $N(a_1^{(i)}, \ldots, a_n^{(i)})$ is a non-degenerate curl.
(2) The limit $Y = \lim_{i\to\infty} X^{(i)}$ is entire.

Note that if (1) holds, the limit Y always exists. That is because for fixed k, l, the entries $x_{k,l}^{(i)}$ of $X^{(i)}$ are non-increasing, but nonnegative. (This is the case even if we allow degenerate curls.) It is clear that curl reductions give rise to curl factorizations.

Lemma 8.1. *Let $X \in U_{\geq 0}$. Then a curl reduction of X exists.*

Proof. Define $X^{(k+1)}$ be applying Lemma 5.3 to $X^{(k)}$: namely,

$$X^{(k+1)} = M(-\epsilon_1(X^{(k)}), -\epsilon_2(X^{(k)}), \ldots, -\epsilon_n(X^{(k)}))X^{(k)}.$$

Let $Y = \lim_{k\to\infty} X^{(k)}$. If Y is finitely supported it is clear that Y is entire, so we assume otherwise, using Lemma 5.2 implicitly in the following. In particular, we assume that the sequence $X^{(k)}$ involves infinitely many non-trivial applications of Lemma 5.3.

We now argue that

$$\frac{x_{i,j}^{(k)}}{x_{i+n,j}^{(k)}} \geq \frac{x_{i,j}^{(k+1)}}{x_{i+n,j}^{(k+1)}}.$$

Indeed, after the substitution $x_{i,j}^{(k+1)} = x_{i,j}^{(k)} - \epsilon_i x_{i+1,j}^{(k)}$ and similarly for $x_{i+n,j}^{(k)}$, the above inequality follows from nonnegativity of the minor of $X^{(k)}$ with rows $i, i+1$ and columns $j-n, j$. We have used $\epsilon_i \neq 0$ for this calculation. We conclude that $\frac{y_{i,j}}{y_{i+n,j}} \leq \frac{x_{i,j}^{(k)}}{x_{i+n,j}^{(k)}}$ for any k. Taking the limit $j \longrightarrow \infty$ we see that $\prod_{i=1}^n \epsilon_i(Y) \leq \prod_{i=1}^n \epsilon_i(X^{(k)})$. We know that the sequence $\frac{1}{r(X^{(k)})} = \prod_{i=1}^n \epsilon_i(X^{(k)})$ is non-increasing as $k \longrightarrow \infty$, but stays nonnegative. Assume its limit δ is non-zero. Then for each k at least one of the $\epsilon_i(X^{(k)})$ is not less than $\delta^{1/n}$. This however would mean that $\sum_k \sum_{j=1}^n a_j^{(k)}$ diverges, which is impossible. Thus $\delta = 0$. Since $\prod_{i=1}^n \epsilon_i(Y)$ is bounded from above by a sequence with zero limit and is nonnegative, it must be the case that $\prod_{i=1}^n \epsilon_i(Y) = 0$. This is equivalent to Y being entire.

We denote by $Z(X) = \prod_{i=1}^\infty N(a_1^{(i)}, \ldots, a_n^{(i)})$ the infinite product obtained from the curl reduction of Lemma 8.1. Such product expressions are called *ASW factorizations* of $Z(X)$.

Proposition 8.2. *Let $X \in U_{\geq 0}$. Then X has a unique curl factorization.*

Proof. Suppose $X = ZY$ is some curl factorization of X. Let us fix a factorization of Z as an infinite product of curls. Since Y is entire, we have $r(Z) \leq r(X)$. Let $N(a_1, a_2, \ldots, a_n)$ be the curl factor in the factorization Z with the smallest radius of convergence, that is, largest value of $\prod_j a_j$. By Lemma 7.9, we have $r(Z) = 1/(\prod_j a_j)$. Using the whirl commutation relations, we may move such a factor to the front of Z, so that $Z = N(a'_1, a'_2, \ldots, a'_n)Z'$ and $\prod_j a'_j = \prod_j a_j$. By Lemma 5.4, we have $a'_i \leq \epsilon_i(X)$ so that $r(Z) \geq r(X)$. It follows that $r(Z) = r(X)$ and $a'_i = \epsilon_i(X)$.

Repeating this argument, we see that the multiset of radii of convergence of curls in Z coincides with that of $Z(X)$. Let $Z^{(k)}$ be the product of the first k curls in the curl reduction of Lemma 8.1, so that $Z(X) = \lim_{k\to\infty} Z^{(k)}$. It is clear that entry-wise $Z^{(k)}$ is less than Z. Let N be arbitrary. Let $b = \min\{\prod_j a_j^{(i)} \mid i \in [1, N)\}$. Pick k so that $Z^{(k)}$ contains all factors in $Z(X)$ with radii of convergence less than or equal to $1/b$. Let Z' be the product of the first N factors in Z. By the whirl commutation relations (Theorem 6.2) we can write $Z^{(k)} = Z'W$ for some $W \in U_{\geq 0}$ – W is obtained by moving to the left all of the factors in Z outside of Z' but with radius of convergence less than or equal to $1/b$. The entries of $Z^{(k)}$ are thus greater than those of Z'. It follows that Z is the limit of the $Z^{(k)}$.

Remark 8.1. In [3] the following question is posed: explicitly describe the transition map between two different factorizations of a totally positive element of $GL_n(\mathbb{R})$. Distinct factorizations of totally positive elements correspond to different double wiring diagrams [11]. Later it was realized [13] that the graph connecting different parametrizations can be completed to a regular graph that is the exchange graph of the corresponding *cluster algebra*. It is natural to ask a similar question in our setting. Let us restrict our attention to infinite products of curls (or whirls). By Proposition 8.2 we have the distinguished ASW factorization, and any other factorization is obtainable by the repeated application of whirl commutation relations. By Corollary 6.4 we can conclude that the graph describing the adjacency between distinct parametrizations of an infinite curl is just the Cayley graph of S_∞ with adjacent transpositions as generators. This graph is already regular and it seems unlikely that analogues of *non-Plücker cluster variables* could arise. The situation becomes more subtle when we allow Chevalley generators in the factorizations. We plan to address these questions in [21].

We call $X \in U$ *doubly entire* if both X and X^{-1} are entire. For TNN matrices, we will usually check the equivalent condition that X and X^{-c} are entire.

Theorem 8.3. *Let $X \in U_{\geq 0}$. Then it has a unique factorization of the form*

$$X = \prod_{i=1}^{\infty} N(a_1^{(i)}, \ldots, a_n^{(i)}) Y \prod_{i=-\infty}^{-1} M(b_1^{(i)}, \ldots, b_n^{(i)}),$$

where all whirls and curls are either non-degenerate or the identity matrix, and the parameters satisfy $\sum_{i,j} a_i^{(i)} + \sum_i b_j^{(i)} < \infty$ and $Y \in U_{\geq 0}$ is doubly entire.

Proof. For existence, first use Proposition 8.2 to write $X = Z(X)X'$ where X' is entire. Now apply Proposition 8.2 to $(X')^{-c}$ to obtain $X' = YW(X)$ where Y^{-c} is entire and $W(X) = \prod_{i=-\infty}^{-1} M(b_1^{(i)}, \ldots, b_n^{(i)})$ and all parameters are positive. We claim that Y is entire. For otherwise, by Lemma 5.3 we can write $X' = N(\epsilon_1(Y), \ldots, \epsilon_n(Y))Y'W(X)$ where $Y'W(X)$ is TNN and the ϵ_i are strictly positive. But X' is entire so this is impossible by Lemma 5.4. Thus $X = Z(X)YW(X)$ is the desired factorization.

For uniqueness, suppose we have a factorization $X = ZYW$ as in the statement of the theorem. By Lemma 7.7, we may apply Proposition 8.2 to $X = Z(YW)$ to see that $Z = Z(X)$. Repeating the argument for X^{-c} we see that $W = W(X)$ is well-defined. (In particular, $W(X)$ can be calculated before or after factoring $Z(X)$ out.)

We call the expression $X = ZYW$ of Theorem 8.3 the *canonical form* of X. We call Z the *curl component* of X and W the *whirl component* of X.

8.2. Doubly entire matrices as exponentials

Lemma 8.4. *Suppose $A(t)$ is doubly entire. Then $A(t) = e^{B(t)}$ for some entire matrix $B(t)$.*

Proof. Define $Z(t) = e^{-\int A^{-1}(t)A'(t)dt}$ where $A'(t)$ denotes $\frac{d}{dt}(A(t))$ and the integral and derivatives are taken entry-wise. Clearly, $Z(t)$ is an entire matrix. We may pick the constant of integration so that $Z(0) = A^{-1}(0)$. This is possible because $A^{-1}(0)$ is non-singular (with inverse $A(0)$). However,

$$\frac{d}{dt}(A(t)Z(t)) = A'(t)Z(t) - A(t)A^{-1}(t)A'(t)Z(t) = 0.$$

Thus $A(t)Z(t)$ is a constant matrix. But $A(0)Z(0)$ is the identity matrix, so the result holds with $B(t) = \int A^{-1}(t)A'(t)dt$ which is clearly entire.

8.3. Infinite products of Chevalley generators

A product of infinitely many non-degenerate whirls (resp. non-degenerate curls) can never be written as a finite product of non-degenerate whirls (resp. non-degenerate curls). This follows from either Lemma 7.2 or the observation that an infinite product of non-degenerate whirls must have infinite support. The situation for Chevalley generators is markedly different. For example, with $n = 2$, one has $\prod_{i=1}^{\infty} M(a_i, 0) = M(\sum_{i=1}^{\infty} a_i, 0)$ assuming that $\sum_{i=1}^{\infty} a_i < \infty$.

Let $S \subset U$ be a subsemigroup of U. We call S a *right limit semigroup* if for all $X^{(1)}, X^{(2)}, \ldots$ in S such that $X = \prod_{i=1}^{\infty} X^{(i)}$ exists, we have $X \in S$. Similarly, we define a *left limit semigroup* by replacing right infinite products with left infinite products.

Let us define the *right Chevalley group* to be the smallest subset $\mathbb{L}_r \subset U_{\geq 0}$ satisfying

(1) every $e_i(a)$ for $a \geq 0$ lies in $\mathbb{L}_r$,
(2) if $X, Y \in \mathbb{L}_r$ then $XY \in \mathbb{L}_r$ (that is, $\mathbb{L}_r$ is a semigroup), and
(3) $\mathbb{L}_r$ is a right limit semigroup.

Note that $\mathbb{L}_r$ exists because we may define $\mathbb{L}_r$ to be the intersection of all (non-smallest) subsets satisfying (1)–(3). We say that $\mathbb{L}_r$ is the right limit semigroup generated by $e_i(a)$. Similarly, we define $\mathbb{L}_l$, the *left Chevalley group* to be the left limit semigroup generated by $e_i(a)$.

Remark 8.2. In [21] we shall show that elements of $\mathbb{L}_r$ (resp. $\mathbb{L}_l$) have "canonical" factorizations.

8.4. Factorization of doubly entire TNN matrices

A TNN matrix $X \in U_{\geq 0}$ is *regular* if it is either (i) the identity matrix, or (ii) doubly entire, infinitely supported and satisfying $\epsilon_i(X) = \mu_i(X) = 0$ for every i. For example, the matrix in Example 2.1 is regular.

Lemma 8.5. *Suppose $X \in U_{\geq 0}$ is entire, and infinitely supported. Then X^{-c} is infinitely supported.*

Proof. Otherwise by Theorem 5.5, X^{-c} is a finite product of possibly degenerate whirls. If X^{-c} is a product of only Chevalley generators then X will be finitely supported, so the factorization of X^{-c} must involve at least one non-degenerate whirl. But then by Lemma 7.2, X would not be entire.

Lemma 8.6. *Suppose X is a doubly-entire infinitely supported TNN matrix. Then $\epsilon_i(X) = 0$ for every i if and only if $\mu_i(X^{-c}) = 0$ for every i.*

Proof. By Lemmas 8.5 and 5.2, X^{-c} is infinitely supported, so $\mu_i(X^{-c})$ is well-defined. By Lemma 5.4, $(\epsilon_1(X), \ldots, \epsilon_n(X))$ records the parameters of the biggest curl which can be factored out of X on the left. Similarly, $(\mu_1(X^{-c}), \ldots, \mu_n(X^{-c}))$ records the parameters of the biggest curl which can be factored out of X^{-c} on the right. Because both X and X^{-c} is entire, such curls are in fact products of Chevalley generators, and inverse of Chevalley generators are Chevalley generators. So we have $\epsilon_i > 0$ for some i, if and only if some Chevalley generator can be factored out on the left of X, if and only if some Chevalley generator can be factored out of X^{-c} on the right, if and only if $\mu_j(X^{-c}) > 0$ for some j.

Corollary 8.7. *A regular matrix $X \in U_{\geq 0}$ satisfies $\epsilon_i(X) = \mu_i(X) = \epsilon_i(X^{-c}) = \mu_i(X^{-c}) = 0$ for every i.*

Theorem 8.8. *Every doubly entire, infinitely supported, $X \in U_{\geq 0}$ can be factorized as $X = AYB$ where $A \in \mathbb{L}_r$, $B \in \mathbb{L}_l$ and $Y \in U_{\geq 0}$ is regular.*

In [21], we shall strengthen Theorem 8.8 by showing that the factorization is unique.

Proof. We use transfinite induction. Every degenerate whirl or curl is a product of Chevalley generators. Pick such a factorization for each degenerate whirl or curl, once and for all.

Now we define a $X_\alpha \in U_{\geq 0}$ for each ordinal α. We define $X_0 = X$. We define $X_{\alpha+1}$ by factoring out a Chevalley generator from X_α on both the left and the right (if possible), always using the first Chevalley generator in the chosen factorization of the curl specified by ASW factorization (Lemma 5.3). If X_α is regular so that no Chevalley generators can be factored out then $X_{\alpha+1} = X_\alpha$. Finally, if α is a limit ordinal, then we set $X_\alpha = \inf_{\beta<\alpha} X_\beta$, where the infimum is taken entry-wise.

If X_α is never regular, then it is easy to see that $\alpha \mapsto X_\alpha$ is injective (X_α is always decreasing). This is impossible because $X_\alpha \in U$, and the cardinality of U is the same as that of the real numbers. Thus X_α is eventually regular, and this is the required matrix Y of Theorem 8.8 (the matrices A and B are obtained by remembering the Chevalley generators used during the transfinite induction).

9. Commuting through infinite whirls and curls

9.1. (Limit) semigroups of infinite whirls and curls

Theorem 9.1. *Each of the sets RC, LC, RW, LW of infinite products of non-degenerate whirls and curls forms a semigroup.*

Example 9.1. Let $n = 2$. Consider the infinite curl $X = \prod_{i \geq 0} N(2^{-i}, 2^{-i-1})$. Then the entries of X for $i < j$ are given by $x_{i,j} = 2^{-\omega(i,j)} \prod_{k=1}^{j-i} \frac{2^k}{2^k - 1}$, where $\omega(i, j) = [(j - i)/2] + 1$ if j is odd and i is even, and $\omega(i, j) = [(j - i)/2]$ otherwise. A fragment of X looks as

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 1 & 2 & \frac{4}{3} & \frac{32}{21} & \frac{256}{315} & \cdots \\ \cdots & 0 & 1 & 1 & \frac{4}{3} & \frac{16}{21} & \cdots \\ \cdots & 0 & 0 & 1 & 2 & \frac{4}{3} & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

One can check using the curl commutation relation that if $a/b = c/d$ then $N(a, b)N(c, d) = N(c, d)N(a, b)$. Using that one verifies that $X^2 = \prod_{i \geq 0} N(2^{-i}, 2^{-i-1})^2$.

We focus on the case of RW. Theorem 9.1 follows from Lemma 9.4.

Lemma 9.2. *Let $X = M(a_1, \ldots, a_n)$ and $Y = M(b_1, \ldots, b_n)$ be two non-degenerate whirls, such that $\max(a_i) < \epsilon < \min(b_i)$. Define $Y' = M(b'_1, \ldots, b'_n)$, $X' = M(a'_1, \ldots, a'_n)$ to be the result of applying the whirl relation, so that $XY = Y'X'$. Then, for each $i \in \{1, 2, \ldots, n\}$*

$$|b'_i - b_i| \leq \epsilon \frac{n \max(b_i)}{\min(b_i)}.$$

Proof. We have

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})} > \frac{\prod_{i=1}^{n} b_i}{\kappa_i(\mathbf{a}, \mathbf{b})} = \frac{b_i}{1 + \sum_{j=i}^{i+n-2} \prod_{k=j+1}^{i+n-1} (a_k/b_k)}$$
$$> b_i(1 - n(a_{i+1}/b_{i+1})) > b_i - \epsilon \frac{n \max_i(b_i)}{\min_i(b_i)}.$$

Similarly, $b_i > b'_i - \epsilon \frac{n \max_i(b_i)}{\min_i(b_i)}$.

Lemma 9.3. *Let $Y_1, Y_2, \ldots, Y_r$ be non-degenerate whirls and $\delta > 0$. Let $X_1, X_2, \ldots, X_m$ be a finite sequence of whirls, and let Y'_j be obtained by successively commuting Y_j through the X_i:*

$$X_1 X_2 \cdots X_m Y_1 Y_2 \cdots Y_r = Y'_1 Y'_2 \cdots Y'_r X'_1 X'_2 \cdots X'_m.$$

Then there is a constant C, depending only on $Y_1, \ldots, Y_r$, such that if the total sum of parameters in all the X_i is less than C, then for each i, the parameters in Y_i differ from those in Y'_i by at most δ.

Proof. Lemma 9.2 allows us to pull the Y-s through the X-s, one after another, guaranteeing that the parameters in the Y_i-s do not change too much. While doing that we need to know that the parameters inside X-s remain small so that we can repeatedly apply Lemma 9.2. This however follows from the fact that parameters in Y-s do not change much, while the total sum of parameters in X-s and Y-s remains constant.

Let $X = \prod_{i=1}^{\infty} X_i$ and $Y = \prod_{i=1}^{\infty} Y_i$ be two infinite products of whirls. We assume the products are written in the canonical ASW order, that is we have $r(X_1^{-c}) \leq r(X_2^{-c}) \leq \cdots$ and similarly for Y. We will call $r(X^{-c})$ the inverse radius of convergence of X. For each $m \in \{1, 2, \ldots\}$, let $X_1, \ldots, X_{s_m}$ and $Y_1, \ldots, Y_{t_m}$ be the factors with inverse radius of convergence less than m. We may rewrite using the ASW factorization

$$X_1 X_2 \cdots X_{s_m} Y_1 Y_2 \cdots Y_{t_m} = Z_1^{(m)} Z_2^{(m)} \cdots Z_{s_m+t_m}^{(m)}.$$

Each of the matrices $Z_1^{(1)}, Z_1^{(2)}, \ldots$ is a whirl depending on n real parameters. These parameters are bounded above by the sum of the parameters in X and Y, so a subsequence of $\{Z_1^{(m_i)}\}$ of converges to some whirl Z_1, which must be non-degenerate. Now find a subsequence of the matrices $\{Z_2^{(m_i)}\}$ which converge to a whirl Z_2, and repeat to define $Z_1, Z_2, \ldots$.

Lemma 9.4. *The infinite product $Z = \prod_{i=1}^{\infty} Z_i$ converges, and $Z = XY$.*

Proof. We first show that $Z_1 Z_2 \cdots Z_k \leq XY$ entrywise. This can be done by finding a sufficiently large m so that $Z_i^{(m)}$ is arbitrarily close to Z_i, for each $i = 1, 2, \ldots, k$, in any desired set of entries. Then we have

$$\prod_{i=1}^{k} Z_i \sim \prod_{i=1}^{k} Z_i^{(m)} \leq \prod_{i=1}^{s_m+t_m} Z_i^{(m)} = X_1 X_2 \cdots X_{s_m} Y_1 Y_2 \cdots Y_{t_m} \leq XY$$

where the inequalities are entrywise.

Conversely, we show that for each j, k, we have $X_1 X_2 \cdots X_j Y_1 Y_2 \cdots Y_k \leq Z$. Pick $r > j$ so large that the sum of all the parameters in $X_r, X_{r+1}, \ldots$ is less than the constant C of Lemma 9.3, for some small $\delta > 0$. Now pick m sufficiently large so that the inverse radius of convergence of $X_1, \ldots, X_r, Y_1, \ldots, Y_k$ are all less than m; in other words, $s_m \geq r$ and $t_m \geq k$. Pick m' sufficiently large so that Z_i and $Z_i^{(m')}$ are arbitrarily close for all $i \leq (s_m + t_m)$. Then

$$\begin{aligned}
\prod_{i=1}^{s_m+t_m} Z_i \sim \prod_{i=1}^{s_m+t_m} Z_i^{(m')} & \\
&= X_1 X_2 \cdots X_{s_m} Y_1' Y_2' \cdots Y_{t_m}' \\
&\geq X_1 X_2 \cdots X_{s_m} Y_1' Y_2' \cdots Y_k' \\
&\sim X_1 X_2 \cdots X_{s_m} Y_1 Y_2 \cdots Y_k \\
&\geq X_1 X_2 \cdots X_j Y_1 Y_2 \cdots Y_k
\end{aligned}$$

where $Y_1', Y_2', \ldots, Y_{t_m}'$ is obtained by commuting $Y_1, Y_2, \ldots, Y_{t_m}$ past $X_{s_m+1}, \ldots, X_{s_{m'}}$. Again the approximations denoted by $\sim$ mean that a finite set of entries is arbitrarily close.

Essentially the same proof establishes a stronger statement. Recall the definition of right and left limit semigroups from Section 8.3.

Theorem 9.5. *The semigroups RC and RW (resp. LC and LW) of infinite products of non-degenerate whirls and curls are right (resp. left) limit semigroups.*

Proof. We prove the statement for RW. Assume we have an infinite product of infinite whirls: $VU\cdots$, each of which has been written in the canonical ASW order. For each $m \in \{1,2,\ldots\}$ denote by $v(m)$ the index such that $V_1,\ldots,V_{v(m)}$ are exactly the whirls in $V = \prod_{i=1}^{\infty} V_i$ with inverse radius of convergence smaller than m. Similarly define $u(m)$ for U, and so on. Note that for each m only finitely many of the factors $V, U, \ldots$ contain a whirl with inverse radius smaller than m. For each m, we define non-degenerate whirls $Z_i^{(m)}$ by the following equality:

$$V_1\cdots V_{v(m)}U_1\cdots U_{u(m)}\cdots W_1\cdots W_{w(m)}X_1\cdots X_{x(m)}Y_1\cdots Y_{y(m)}$$
$$= Z_1^{(m)}\cdots Z_{v(m)+\cdots+y(m)}^{(m)}$$

where the $Z_i^{(m)}$ are in the canonical ASW order.

As before the proof of Lemma 9.4, choose subsequences of m-s to define $Z_1, Z_2, \ldots$. We now claim that $VU\cdots = Z_1Z_2\cdots$. For the inequality $Z_1\cdots Z_q \le VU\cdots$, the proof is the same as in Lemma 9.4. For the other direction let us assume we are given a product

$$V_1\cdots V_vU_1\cdots U_u\cdots W_1\cdots W_wX_1\cdots X_xY_1\cdots Y_y$$

of initial parts of certain finite number of initial factors. We now repeatedly apply Lemma 9.3, in a similar manner to the proof of Lemma 9.4. Namely, choose m_1 so that

$$Y_1\cdots Y_y \sim Y_1'\cdots Y_y',$$

where $Y_1'\cdots Y_y'$ is obtained by commuting $Y_1\cdots Y_y$ through $X_{x(m_1)+1}\cdots X_N$ for some N. We may assume that $x(m_1) > x$. By Lemma 9.3 we may assume m_1 is chosen so that the approximation holds for any N.

Similarly choose $m_2 > m_1$ so that

$$X_1\cdots X_{x(m_1)}Y_1'\ldots Y_y' \sim X_1'\cdots X_{x(m_1)}'Y_1''\ldots Y_y''.$$

Here $X_1'\cdots X_{x(m_1)}'Y_1''\ldots Y_y''$ is obtained by pulling $X_1\cdots X_{x(m_1)}Y_1'\ldots Y_y'$ through the product $W_{w(m_2)+1}\cdots W_N$. Again we assume that $w(m_2) > w$. On the next step we find $m_3 > m_2$ that would allow to pull

$$W_1\cdots W_{w(m_2)}X_1'\cdots X_{x(m_1)}'Y_1''\ldots Y_y''$$

through the next factor, and so on. Finally let $m = \max(m_i)$ be the parameter in the last move and find m' so that $\prod_{i=1}^{v(m)+\cdots+y(m)} Z_i^{(m')}$ is arbitrarily close to $\prod_{i=1}^{v(m)+\cdots+y(m)} Z_i$. Now we calculate

$$\begin{aligned}\prod_{i=1}^{v(m)+\cdots+y(m)} Z_i \sim \prod_{i=1}^{v(m)+\cdots+y(m)} Z_i^{(m')} &= V_1\cdots V_{v(m)}\cdots W_1^*\cdots W_{w(m_2)}^*X_1^*\cdots X_{x(m_1)}^*Y_1^*\cdots Y_y^*A\\ &\ge V_1\cdots V_{v(m)}\cdots W_1^*\cdots W_{w(m_2)}^*X_1^*\cdots X_{x(m_1)}^*Y_1^*\cdots Y_y^*\\ &\sim V_1\cdots V_{v(m)}\cdots W_1\cdots W_{w(m_2)}X_1\cdots X_{x(m_1)}Y_1\cdots Y_y\\ &\ge V_1\cdots V_v\cdots W_1\cdots W_wX_1\cdots X_xY_1\cdots Y_y.\end{aligned}$$

We explain the equality on the second line. Here W_i^*, X_i^*, Y_i^* denote what we get when we commute $Y_1\cdots Y_y$ past $X_{x(m_1)+1}\cdots X_{x(m')}$, and then commute $X_1\cdots X_{x(m_1)}Y_1\cdots Y_y$ past

$W_{w(m_2)+1} \cdots W_{w(m')}$, and so on. Applying to $\prod_{i=1}^{v(m')} V_i \cdots \prod_{i=1}^{w(m')} W_i \prod_{i=1}^{x(m')} X_i \prod_{i=1}^{y(m')} Y_i$ all these commutations we obtain $V_1 \cdots V_{v(m)} \cdots X_1^* \cdots X_{x(m_1)}^* Y_1^* \cdots Y_y^* B$ where B consists of the whirls obtained from

$$V_{v(m)+1}, \dots, V_{v(m')}, \dots, W_{w(m_2)+1}, \dots, W_{w(m')}, X_{x(m_1)+1}, \dots,$$
$$X_{x(m')}, Y_{y+1}, \dots, Y_{y(m')}$$

via commutation. The matrix A is what we get when we in addition commute all the whirls in B with inverse radius of convergence greater than m to the right and remove them.

9.2. Chevalley generators out of whirls

We have shown that RC, LC, RW, and LW are semigroups. We now describe what happens when they are multiplied by Chevalley generators from a particular side. We only state our results for right-infinite whirls and curls.

Theorem 9.6. *Suppose $e_i(a)$ is a Chevalley generator and $X \in RW$ (resp. $X \in RC$). Then $e_i(a)X \in RW$ (resp. $e_i(a)X \in RC$).*

Example 9.2. If X is the right-infinite curl in Example 9.1 then

$$\begin{aligned} e_1(1)X &= N\left(2, \frac{1}{4}\right) N\left(\frac{1}{4}, \frac{1}{2}\right) N\left(\frac{1}{2}, \frac{1}{16}\right) N\left(\frac{1}{16}, \frac{1}{8}\right) \cdots \\ &= \prod_{i \geq 0} N(2^{1-2i}, 2^{-2-2i}) N(2^{-2-2i}, 2^{-1-2i}). \end{aligned}$$

Theorem 9.6 follows from the following more precise lemma.

Lemma 9.7. *Let $e_i(a)$ be a Chevalley generator and $X = \prod_{k=1}^{\infty} M(b_1^{(k)}, \dots, b_n^{(k)})$ be a right-infinite product of non-degenerate whirls. Using the whirl commutation relation of Theorem 6.2, we define $\mathbf{c}^{(k)} = (c_1^{(k)}, \dots, c_n^{(k)})$ and $a^{(j)}$ by*

$$e_i(a) \prod_{k=1}^{\infty} M(\mathbf{b}^{(k)}) = \prod_{k=1}^{j} M(\mathbf{c}^{(k)}) e_{i-j}(a^{(j)}) \prod_{k=j+1}^{\infty} M(\mathbf{b}^{(k)}). \tag{3}$$

Then

(1) $\lim_{j \to \infty} a^{(j)} = 0$.
(2) *The product $\prod_{k=1}^{\infty} M(\mathbf{c}^{(k)})$ is well-defined and equals X.*

The analogous statement holds for curls.

We may think of Lemma 9.7 as saying that infinite products $\prod_{i=1}^{\infty} M_i^{(j)}$ of whirls (or curls) “absorb” Chevalley generators (if multiplied on the correct side).

Proof. In the setting of Lemma 6.7 one has $a'_{i+1} = \frac{b_{i+1} a_i}{a_i + b_i} < b_{i+1}$. In order for the product X to be well-defined it must be the case that $\lim_{j \to \infty} b_{i-j}^{(j)} = 0$, and so $\lim_{j \to \infty} a^{(j)} = 0$, proving the first statement.

For the second part, consider a fixed entry $x_{s,t}$. Suppose that the sequence $X_j = e_i(a) \prod_{k=1}^{j} M(b_1^{(k)}, \dots, b_n^{(k)})$ of matrices has entries m_j in location (s,t). Then $\lim_{j \to \infty} m_j = x_{s,t}$.

Similarly define m'_j as the corresponding entry of $X'_j = \prod_{k=1}^{j} M(c_1^{(k)}, \dots, c_n^{(k)})$. Clearly $\lim_{j\to\infty} m'_j$ exists and is less than $x_{s,t}$. We must show that the limit equals $x_{s,t}$.

For a given $\delta > 0$ one can choose j large enough so that $a^{(j)} x_{s,t-1} < \delta/2$ and $x_{s,t} - m_j < \delta/2$. The equality (3) shows that $m_j - m'_j \le a^{(j)} x_{s,t-1}$, so we deduce that $x_{s,t} - m'_j < \delta$. Thus $\lim_{j\to\infty} m'_j = x_{s,t}$.

The proof for curls is verbatim, using the inequality $a'_{i-1} = \frac{b_{i-1} a_i}{a_i + b_i} < b_{i-1}$ from Lemma 6.8.

9.3. Not all Chevalley generators at once

The ϵ-sequence of a TNN matrix X gives a bound on what Chevalley generators can be factored out from X on the left so that the result remains TNN. In particular, by Lemma 5.4, $e_i(a)$ cannot be factored out if $a > \epsilon_i$. This bound is far from sharp: for example no Chevalley generator can be factored out from a non-degenerate curl, but every ϵ_i of a curl is strictly positive.

Proposition 9.8. *Let $X \in U_{\ge 0}$. There is an $i \in \mathbb{Z}/n\mathbb{Z}$ such that if $X = e_i(a) X'$ for $a \ge 0$ and $X' \in U_{\ge 0}$ then $a = 0$.*

Proof. Assume the statement is false and that for each j we have $X = e_j(a_j) X_j$ for some TNN X_j-s and $a_j > 0$. By Theorem 8.3 one can write $X_j = \prod_{i=1}^{\infty} N_i^{(j)} E^{(j)}$ where $E^{(j)}$ is entire. There are two cases to consider.

Case (1). One of the products $\prod_{i=1}^{\infty} N_i^{(j)}$ has only finitely many non-trivial terms. Then one can commute $e_j(a_j)$ through this product to obtain another finite product of curls times $e_{j'}(a_{j'}) E^{(j)}$, which is entire. Since the decomposition of Theorem 8.3 is unique, this means by Lemma 9.7 that the products $\prod_{i=1}^{\infty} N_i^{(j)}$ are finite for each $j \in \mathbb{Z}/n\mathbb{Z}$ and that the corresponding expressions $e_{j'}(a_{j'}) E^{(j)}$ are all equal to some entire matrix E (what we get from X by removing the curl component of X). As j varies over $\mathbb{Z}/n\mathbb{Z}$, so does j'. Furthermore, each $a_{j'} > 0$. This is impossible, because E, being entire, has one of the ϵ-s equal to 0, and the corresponding Chevalley generator cannot be factored out with any positive constant.

Case (2). All the products $\prod_{i=1}^{\infty} N_i^{(j)}$ are infinite. Let $X = \prod_{i=1}^{\infty} N_i E$ factorize X into its curl component and an entire matrix. By Lemma 9.7 and by the uniqueness in Theorem 8.3 we have $\prod_{i=1}^{\infty} N_i = e_j(a_j) \prod_{i=1}^{\infty} N_i^{(j)}$ for every j. Without loss of generality we can assume that each such product of curls is an ASW factorization.

Let us consider what happens to the a_j when we commute $e_j(a_j)$ past $N_1^{(j)} = N(\mathbf{b}^{(j)})$. We know that $e_j(a_j) N(\mathbf{b}^{(j)}) = N(\mathbf{b}') e_{j'}(a'_j)$, where $N_1 = N(\mathbf{b}')$ does not depend on j. We calculate using Lemma 6.8 that $a'_{j-1}/a_j = b'_{j-1}/(b'_j - a_j) > b'_{j-1}/b'_j$. Note that there are no references to $\mathbf{b}^{(j)}$ in these inequalities.

Now we observe that

$$\frac{\prod_{j\in\mathbb{Z}/n\mathbb{Z}} a'_j}{\prod_{j\in\mathbb{Z}/n\mathbb{Z}} a_i} = \prod_{j\in\mathbb{Z}/n\mathbb{Z}} \frac{a'_{j-1}}{a_j} > \prod_{j\in\mathbb{Z}/n\mathbb{Z}} \frac{b'_{j-1}}{b'_j} = 1.$$

Thus the total product of parameters in the $e_j(a_j)$ increases after commuting past $N_1^{(j)}$. The same argument shows that the product of parameters will continue to increase as we commute past $N_2^{(j)}, N_3^{(j)}, \dots$. This contradicts Lemma 9.7, which says that all n Chevalley parameters have zero limit.

9.4. Pure whirls and curls

Let us call $X \in RW \cap LW$ a *pure whirl*, and write $PW = RW \cap LW$. Similarly we define the set PC of *pure curls*. By Theorem 9.1, we have

Example 9.3. The right-infinite curl X from Example 9.1 is pure. Indeed, one can see that X has southwest–northeast axes of symmetry, and thus its factors could be multiplied in the reverse direction: $X = \prod_{i=-\infty}^{0} N(2^{-i}, 2^{-i-1})$. One can also derive this from the fact that the curl factors in X commute.

Theorem 9.9. *The sets PW and PC of pure whirls and curls are semigroups.*

Certain properties of pure whirls and curls are immediately clear, for example it follows from Lemma 7.7 that elements of PW have all ϵ_i-s and μ_i-s equal to 0. We state the following result only for infinite curls. The result for whirls is obtained by applying $^{-c}$.

Theorem 9.10. *Each $X \in RC$ can be uniquely factored as $X = EX'$, where E is doubly-entire and $X' \in PC$. Similarly, each $X \in LC$ can be uniquely factored as $X = X'E$, where E is doubly-entire and $X' \in PC$.*

Proof. We consider the case of LC, the case of RC being identical. Apply Theorem 8.3 to obtain $X = X'E$ where E is entire and $X' \in RC$. The matrix E must be doubly-entire, for otherwise a non-degenerate whirl can be factored out of X on the right. But this would mean that a non-degenerate curl can be factored out of X^{-c} on the left. This is impossible by Lemma 7.7, since X^{-c} is an infinite product of whirls.

The factorization $X = X'E$ is unique, so it remains to show that $X' \in PC$. Apply (left–right swapped) Theorem 8.3 to X' to rewrite it as $X' = FX''$, where $X'' \in LC$ and F is entire. Finally, rewrite $X''E$ as GX''' where $X''' \in LC$ and G is entire. In the end we get $X = FGX'''$. By Theorem 8.3 and the assumption that $X \in LC$, the entire matrix GF must be trivial, and thus F is trivial. This means exactly $X' \in LC$.

10. Minor ratio limits

10.1. Ratio limit interpretation and the factorization problem

Let $X \in RC$ and let $X = \prod_{i=1}^{\infty} N(a_1^{(i)}, \dots, a_n^{(i)})$ be the ASW factorization of X. Let $k \geq 1$ be an integer. Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be a collection of positive integers such that $i_t \leq i+t$ for an integer i, and let $I_i^k = \{i+1, i+2, i+3, \dots, i+k\}$. Also let $J_h^k = \{h+1, h+2, \dots, h+k\}$. By Theorem 7.4, the minor $\Delta_{I,J_j^k}(X)$ is equal to $s_{\lambda_{I,h}^k}(\mathbf{a})$, where $\lambda_{I,h}^k = \lambda(I, J_h^k)$ is a skew shape the right-hand side of which is vertical. We also define $\mu_{i,h}^k = \lambda(I_i^k, J_h^k)$, which is a rectangular skew shape of height k and width $h-i$.

It is clear that $\mu_{i,j}^k \subset \lambda_{I,h}^k$. We let $\nu = (i+k, \dots, i+k)/(i_k, \dots, i_1+k-1)$ be the difference. We define

$$s_\nu(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(k)}) = \sum_T a^T$$

to be the weight generating function of tableaux with shape ν, and filled with numbers from $[1,k]$. We can also obtain $s_\nu(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(k)})$ from $s_\nu(\mathbf{a})$ by setting $a_j^{(i)} = 0$ for $i > k$. For example, for $I = (i, i+2, \dots, i+k)$ we have $s_\nu(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \dots, \mathbf{a}^{(k)}) = \sum_{j=1}^{k} a_i^{(j)}$.

Theorem 10.1. *We have*

$$s_\nu(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}) = \lim_{h\to\infty} \frac{s_{\lambda^k_{I,h}}(\mathbf{a})}{s_{\mu^k_{i,h}}(\mathbf{a})} = \lim_{h\to\infty} \frac{\Delta_{I,J^k_h}(X)}{\Delta_{I^k_i,J^k_h}(X)}.$$

Proof. The general plan is similar to the proof of Lemma 7.8, but the details are significantly more complicated.

We can write

$$\frac{s_{\lambda^k_{I,h}}(\mathbf{a})}{s_{\mu^k_{i,h}}(\mathbf{a})} = s_\nu(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(k)}) + \frac{\mathrm{wt}(S^k_h)}{\Delta_{I^k_i,J^k_h}(X)},$$

where S^k_h is the set of all semistandard fillings of $\lambda^k_{I,h}$ such that not all numbers filling the left ν part of the shape are in the range from 1 to k. It remains to show that $\lim_{h\to\infty} \frac{\mathrm{wt}(S^k_h)}{\Delta_{I^k_i,J^k_h}(X)} = 0$. Denote by T^k_h the set of all semistandard fillings of $\mu^k_{i,h}$ with entries in the bottom row not smaller than $k+1$. Since $\mathrm{wt}(S^k_h) < s_\nu(\mathbf{a})\mathrm{wt}(T^k_h)$, it suffices to show that

$$\lim_{h\to\infty} \frac{\mathrm{wt}(T^k_h)}{\Delta_{I^k_i,J^k_h}(X)} = 0.$$

We shall prove by induction a stronger statement. Namely, let us take a vector $b = (b_1, \ldots, b_{h+k})$ of positive integers we call *bounds*. We also allow some of the bounds to be infinite. Denote by $T^{k,b}_h$ the set of all semistandard tableaux of shape $\mu^k_{i,h}$ with entries in the bottom row not smaller than $k+1$ and smaller than the corresponding entries of b (that is, an entry in the r-th column has to be less than or equal to b_r). One can think of b as a hidden $(k+1)$-st row of the tableau. Similarly denote by $U^{k,b}_h$ the set of all semistandard fillings of $\mu^k_{i,h}$ with the first entry in the bottom row equal k and all entries in the bottom row less than the corresponding entry of b. Let $V^{k,b}_h = U^{k,b}_h \cup T^{k,b}_h$ be the set of all semistandard fillings of $\mu^k_{i,h}$ with entries in the bottom row less than the corresponding entry of b. We claim that for a fixed $\varepsilon > 0$ there is N such that for $h \geq N$ we have $\mathrm{wt}(T^{k,b}_h)/\mathrm{wt}(U^{k,b}_h) < \varepsilon$, or equivalently $\mathrm{wt}(V^{k,b}_h)/\mathrm{wt}(U^{k,b}_h) < 1+\varepsilon$ for any b such that $V^{k,b}_h$ (and thus $U^{k,b}_h$) is non-empty.

We proceed by induction on $k \geq 0$. Checking the base case $k = 1$ is essentially the same as checking the inductive step, so assume now that the claim has been proved for the values up to k, and prove it for the $(k+1)$-row case. By the induction assumption, for any ε there exists an N such that for $h \geq N$ and any bound b', the fillings of the first k rows with the first column filled with the numbers $1, \ldots, k$ constitute at least $1/(1+\varepsilon)$ part of weight of all possible fillings. Iterating, we can claim that for any m and ε there exists an N such that for $h \geq N$ and any bound b' the fillings of the first k rows with the first m columns filled minimally constitute at least $1/(1+\varepsilon)$ portion of weight of all possible fillings. Thinking of the bounds b' as a $(k+1)$-st row, we now sum over all b' which are compatible with given bound b, and conclude that for any m and ϵ there is an N such that for $h \geq N$

$$\frac{\mathrm{wt}(V^{k+1,b}_h)}{\mathrm{wt}(W^{k+1,b}_{m,h})} < 1+\varepsilon.$$

Here $\mathrm{wt}(W_{m,h}^{k+1,b})$ denotes all fillings of $\mu_{i,h}^{k+1}$ compatible with b such that the rectangle formed by first k rows and first m columns is filled minimally, that is, with the numbers $1, 2, \ldots, k$. Now among let $T_{m,h}^{k+1,b} \subset W_{m,h}^{k+1,b}$ be the subset of tableaux with the lowest row filled with numbers greater than $k+1$, and the $U_{m,h}^{k+1,b} \subset W_{m,h}^{k+1,b}$ be the subset of tableaux with the lower left corner filled with $k+1$. Note that $\mathrm{wt}(U_{m,h}^{k+1,b}) < \mathrm{wt}(U_{h}^{k+1,b})$ since dropping the minimality condition on the second to m-th rows can only increase the sum.

Pick R so that

$$\sum_{\ell \geq R} \sum_{j=1}^{n} a_j^{(\ell)} < \min(a_1^{(k+1)} a_2^{(k+1)}, \ldots, a_n^{(k+1)}).$$

This can be done since the sum of all $a_j^{(\ell)}$ is finite. Let $Q_{m,h}^{k+1,b} \subset T_{m,h}^{k+1,b}$ be the subset of tableaux with only the labels $1, 2, \ldots, R$ in the first m columns. We define a map $T_{m,h}^{k+1,b} \to Q_{m,h}^{k+1,b}$ by changing every entry in the last row and first m columns which is greater than R, to R. As we did in Lemma 7.8, we give tableaux in $Q_{m,h}^{k+1,b}$ a modified weight, denoted wt': the entries in a cell with residue j, in the last row and first m columns, labeled R, have weight equal to $a_j^{(k+1)}$. All the other entries have the usual weight. By our choice of R, we have $\mathrm{wt}(T_{m,h}^{k+1,b}) < \mathrm{wt}'(Q_{m,h}^{k+1,b})$.

For any $T \in Q_{m,h}^{k+1,b}$, there is some $r \in [k+1, R]$ such that there are at least m/R cells filled with r in the last row. If there are several options for r choose the smallest one. Let us change the last row by removing the first $n, 2n, \ldots,$ of the r's, changing them to $(k+1)$'s placed in the front of the row. As a result we get a filling that agrees with the bound b since the entry of each cell did not increase. This produces m/Rn distinct tableaux in $U_h^{k+1,b}$. The weight of the resulting tableau is at least as large as the modified weight of the original one: if $r < R$ this follows from the fact that in an ASW factorization the products of parameters in successive curls do not increase. If $r = R$ this follows by definition of the modified weight.

Thus we obtain a multi-valued map from $Q_{m,h}^{k+1,b}$ to $U_{m,h}^{k+1,b}$ such that each element of $Q_{m,h}^{k+1,b}$ maps into m/Rn elements of $U_{m,h}^{k+1,b}$, while each element of $U_{m,h}^{k+1,b}$ is the image of less than R elements of $T_{m,h}^{k+1,b}$. Thus we have

$$\mathrm{wt}(T_{m,h}^{k+1,b}) < \frac{m}{Rn} \mathrm{wt}'(Q_{m,h}^{k+1,b}) < R \ \mathrm{wt}(U_{m,h}^{k+1,b}),$$

which implies

$$\mathrm{wt}(W_{m,h}^{k+1,b}) < \left(1 + \frac{R^2 n}{m}\right) \mathrm{wt}(U_{m,h}^{k+1,b}).$$

Now we can combine several claims to get

$$\begin{aligned} \mathrm{wt}(V_h^{k+1,b}) &< (1+\varepsilon)\mathrm{wt}(W_{m,h}^{k+1,b}) < (1+\varepsilon)\left(1 + \frac{R^2 n}{m}\right) \mathrm{wt}(U_{m,h}^{k+1,b}) \\ &\leq (1+\varepsilon)\left(1 + \frac{R^2 n}{m}\right) \mathrm{wt}(U_h^{k+1,b}). \end{aligned}$$

Clearly for any $\delta > 0$ one can choose $\epsilon > 0$ and large enough m so that $(1+\epsilon)(1+\frac{R^2 n}{m}) < 1+\delta$, which finishes the proof.

Remark 10.1. In [21], we shall give a different interpretation of limit ratio minors for arbitrary TNN matrices, not just for infinite products of curls.

Example 10.1. The definition of ϵ_i as the limit $\lim_{j\to\infty} \frac{x_{i,j}}{x_{i+1,j}}$ is an instance of Theorem 10.1 with $k=1$ and $I=\{i\}$.

Example 10.2. Take the matrix from Example 5.1. Take $i=2$, $k=2$, and $I=(1,4)$. Then

$$\lim_{h\to\infty} \frac{\Delta_{I,J_h^k}(X)}{\Delta_{I_i^k,J_h^k}(X)} = \lim_{g\to\infty} \frac{\det\begin{pmatrix} 2^{g+2}-3 & 2^{g+2}-2 \\ 3\cdot 2^{g-1}-3 & 3\cdot 2^{g-1}-2 \end{pmatrix}}{\det\begin{pmatrix} 2^{g+1}-3 & 2^{g+1}-2 \\ 3\cdot 2^{g-1}-3 & 3\cdot 2^{g-1}-2 \end{pmatrix}} = 5.$$

And indeed, this is the value of $s_{(4,4)/(4,2)} = h_2^{(1)}$ evaluated at the first (and in this case—the only) two curls of the ASW factorization: $\frac{4}{3}\cdot\frac{3}{2}+\frac{4}{3}\cdot\frac{3}{2}+\frac{2}{3}\cdot\frac{3}{2}=5$.

The proof of Theorem 10.1 clearly works for X a finite product of curls as long as k is not larger than the number of curls in the product. The following immediate corollary allows to express all the parameters involved in the ASW factorization of an infinite curl directly through the minor ratio limits.

Corollary 10.2. *We have*

$$a_i^{(k)} = \lim_{h\to\infty} \frac{\Delta_{I_{i-1}^k,J_h^k}(X)}{\Delta_{I_i^k,J_h^k}(X)} \Big/ \lim_{h\to\infty} \frac{\Delta_{I_i^{k-1},J_h^{k-1}}(X)}{\Delta_{I_{i+1}^{k-1},J_h^{k-1}}(X)}.$$

Proof. By Theorem 10.1 the numerator is equal to $\prod_{j=i+k-1}^{i} a_j^{(i+k-j)}$ and the denominator is equal to $\prod_{j=i+k-1}^{i+1} a_j^{(i+k-j)}$, from which the statement follows.

It appears that even in the case $n=1$ the result of Theorem 10.1 is new, we state it separately as follows. Let $\mathbf{a}=a_1,a_2,\ldots$ be a sequence of parameters such that $\sum_i a_i < \infty$ and let s_λ denote the usual Schur function. Let $\nu=(i+k,\ldots,i+k)/(i_k,\ldots,i_1+k-1)$ and adopt other notation as above.

Corollary 10.3. *The limit of ratios of Schur functions* $\lim_{h\to\infty} s_{\lambda_{I,h}^k}(\mathbf{a})/s_{\mu_{i,h}^k}(\mathbf{a})$ *is equal to the Schur polynomial* $s_\nu(a_{i_1},a_{i_2},\ldots,a_{i_k})$ *evaluated at the* k *largest parameters among the* a_i*-s.*

10.2. Invariance

In [35] to any non-crossing matching τ on $2n$ vertices and to any permutation $w\in S_n$ a number $f_\tau(w)$ was associated using the Temperley–Lieb algebra. Let $Y=(y_{st})$ be an $n\times n$ matrix variables. One can then construct a family of polynomials

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(Y) := \sum_{w\in S_n} f_\tau(w)\, y_{1,w(1)}\cdots y_{n,w(n)}$$

called *Temperley–Lieb immanants*. Let us consider $2n$ points $\{1,2,\ldots,2n\}$ arranged in two columns, with the numbers $\{1,2,\ldots,n\}$ arranged from top to bottom in the left column, and the numbers $\{n+1,n+2,\ldots,2n\}$ arranged from top to bottom in the right column. A (complete)

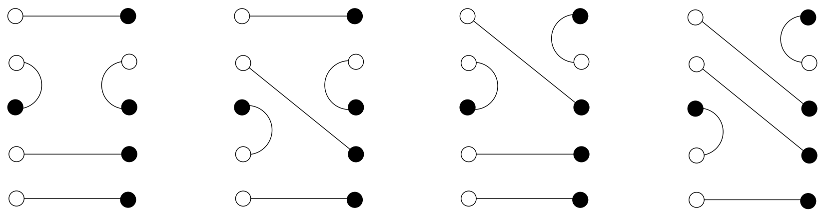

Fig. 7. An example of S-compatible non-crossing matchings.

matching of $[2n]$ is called non-crossing if it can be drawn without intersecting edges, where edges are not allowed to leave the rectangle bounded by $1, n, n+1, 2n$. For a subset $S \subset [2n]$, let us say that a non-crossing (complete) matching is *S-compatible* if each strand of the matching has one endpoint in S and the other endpoint in its complement $[2n] \setminus S$. Coloring vertices in S black and the remaining vertices white, a non-crossing matching is S-compatible if and only if each edge in it has endpoints of different color. Let $\Theta(S)$ denote the set of all S-compatible non-crossing matchings. An example for $n = 5$, $S = \{3, 6, 7, 8, 10\}$ is shown in Fig. 7. For a subset $I \subset [n]$ let $\bar{I} := [n] \setminus I$ and let $I^{\wedge} := \{2n + 1 - i \mid i \in I\}$. The following results were obtained in [35].

Theorem 10.4 (*[35, Propositions 2.3, 4.4]*). *If Y is a totally nonnegative matrix, then $\mathrm{Imm}_\tau^{\mathrm{TL}}(Y) \geq 0$. For two subsets $I, J \subset [n]$ of the same cardinality and $S = J \cup (\bar{I})^{\wedge}$, we have*

$$\Delta_{I,J}(Y) \cdot \Delta_{\bar{I},\bar{J}}(Y) = \sum_{\tau \in \Theta(S)} \mathrm{Imm}_\tau^{\mathrm{TL}}(Y).$$

Let now $I = i_1 < i_2 < \cdots < i_k$, $I' = i'_1 < i'_2 < \cdots < i'_k$, $J = j_1 < j_2 < \cdots < j_k$ and $J' = j'_1 < j'_2 < \cdots < j'_k$ be four k-tuples of positive integers such that $i_t \leq i'_t$ and $j_t \leq j'_t$ for each$1 \leq t \leq k$.

Lemma 10.5. *For a totally nonnegative matrix X we have*

$$\frac{\Delta_{I,J}(X)}{\Delta_{I',J}(X)} \geq \frac{\Delta_{I,J'}(X)}{\Delta_{I',J'}(X)}$$

as long as the denominators are non-zero.

Proof. Let Y be the $2k \times 2k$ submatrix of X induced by the rows in $I \cup I'$ and columns $J \cup J'$, where we repeat a row or a column if it belongs to both of the sets (that is, $I \cup I'$ and $J \cup J'$ are considered multisets). We index rows and columns of Y again by $I \cup I'$ and $J \cup J'$. Whenever there is a repeated column we consider the right one of the two to be in J'. Similarly whenever there is a repeated row we consider the bottom one of the two to be in I'. Then $I' = \bar{I}$, $J' = \bar{J}$ and we can apply the above theorem to the products $\Delta_{I,J}(X)\Delta_{I',J'}(X)$ and $\Delta_{I',J}(X)\Delta_{I,J'}(X)$.

The coloring of $4k$ points one obtains from I and J' (that is, $S = J' \cup (\bar{I})^{\wedge}$) has the property that both in the right and left columns there are more white vertices near the top than black vertices. More precisely, the t-th white vertex is above the t-th black vertex. This follows from the conditions $i_t \leq i'_t$ and $j_t \leq j'_t$. Its easy to see this property of the coloring implies that all edges of a compatible matching have either both endpoints on the left or both on the right. Indeed, if there is an edge connecting the two sides then the non-crossing condition implies that the vertices above its endpoints on either side should have an equal number of black and white vertices. This contradicts "more white vertices near the top".

The coloring coming from I and J (that is, $S = J \cup (\bar{I})^\wedge$) is obtained by swapping black and white colors on the left. It follows that every compatible matching remains compatible. Thus every Temperley–Lieb immanant occurring in the decomposition of $\Delta_{I',J}(X)\Delta_{I,J'}(X)$ occurs also in the decomposition of $\Delta_{I,J}(X)\Delta_{I',J'}(X)$. Since immanants are nonnegative, by Theorem 10.4 we conclude that

$$\Delta_{I,J}(X)\Delta_{I',J'}(X) - \Delta_{I',J}(X)\Delta_{I,J'}(X) \geq 0.$$

Call a sequence $J^s = j_1^s < j_2^s < \cdots < j_k^s$ increasing if $j_t^s < j_t^{s+1}$ for any t and s.

Theorem 10.6. *For a totally positive matrix X, I and I' as above and any increasing sequence J^s the limit*

$$\lim_{s\to\infty} \frac{\Delta_{I,J^s}(X)}{\Delta_{I',J^s}(X)}$$

exists and does not depend on the choice of the sequence J^s.

Proof. The fact that the limit exists follows from Lemma 10.5: the ratio is non-increasing and remains nonnegative. To see that it is independent of J^s, assume there is another sequence J'^s. Then for every element $J^s = j_1^s < j_2^s < \cdots < j_k^s$ there is an element $J'^t = j_1'^t < j_2'^t < \cdots < j_k'^t$ such that $j_r^s < j_r'^s$ for every r. This means that

$$\lim_{s\to\infty} \frac{\Delta_{I,J^s}(X)}{\Delta_{I',J^s}(X)} \leq \lim_{s\to\infty} \frac{\Delta_{I,J'^s}(X)}{\Delta_{I',J'^s}(X)}.$$

However in the same way we obtain the backwards inequality. Thus the two limits are equal.

11. Some open problems

We collect here some questions and conjectures.

From Section 3.

Conjecture 11.1. *Corollary* 3.6 *holds for all $X \in GL_n(\mathbb{R}((t)))_{\geq 0}$.*

Question 11.2. Can *every* TNN matrix be represented by a possibly infinite, not necessarily acyclic cylindric "network"?

From Section 4.

Problem 11.3. Let X be a TNN matrix. Then every entry $\overline{x}_{ij}(t)$ of $\overline{X}(t)$ is a totally positive function. What is the relationship between the poles and zeros (see Theorem 1.2) of different entries $\overline{x}_{ij}(t)$?

Problem 11.4. Brenti [5] has studied combinatorics of Polya frequency sequences, as well as generalizations such as log-concave sequences. Can his questions and results be generalized to $n > 1$?

From Section 6. The following problem is inspired by [3].

Problem 11.5. Let $w \in S_\infty$ be applied to an infinite curl via the maps η_i, as in Corollary 6.4. Describe the parameters of the resulting product explicitly as rational functions of the original parameters.

Example 11.1. For $n = 3$ applying $w = s_1s_2s_1$ to $N(\mathbf{a})N(\mathbf{b})N(\mathbf{c})\cdots$ we get $c_1' =$

$$\frac{c_3(a_1a_3b_1b_2 + a_1b_2b_3c_1 + a_1a_3b_2c_1 + b_1c_1c_2c_3 + b_1b_3c_1c_2 + a_1a_3c_1c_2 + a_1c_1c_2c_3 + a_1b_3c_1c_2 + c_1^2c_2c_3)}{a_3b_1b_2c_3 + b_2b_3c_1c_3 + a_3b_2c_1c_3 + a_2a_3b_1c_3 + a_2a_3b_1b_3 + a_2a_3c_1c_3 + c_1c_2c_3^2 + b_3c_1c_2c_3 + a_3c_1c_2c_3}.$$

From Section 7.

Problem 11.6. Suppose $X = \prod_{i=1}^\infty N(\mathbf{a}^{(i)})$ is an infinite product of curls (or whirls). Each entry of $\overline{X}(t)$ is a totally positive function. What can we say about the poles and zeros of $\overline{x}_{i,j}(t)$, in terms of the parameters $\mathbf{a}^{(i)}$?

From Section 8. The following problem is non-trivial even when $Y = Y'$ is the identity matrix.

Problem 11.7 (*Multiplication of Canonical Forms*)**.** Let $X = ZYW$ and $X' = Z'Y'W'$ be written in canonical form. How can one write XX' in canonical form?

Problem 11.8. Repeat Problem 11.6 for matrices in canonical form.

From Section 9. One can break Problem 11.7 into smaller more specific problems.

Question 11.9 (*Commutation of Infinite Whirls with Infinite Curls*)**.** Let $X \in RC$ (or LC) and $Y \in RW$ (or LW). When is it possible to write XY as $Y'X'$, where $Y' \in RW$ (or LW) and $X' \in RC$ (or LC)?

Question 11.10 (*Product of Opposing Whirls or Curls*)**.** Let $X \in RC$ (resp. RW) and $Y \in LC$ (resp. LW). How does one rewrite XY in canonical form?

Question 11.11 (*Doubly-Infinite Whirls or Curls*)**.** How does one rewrite in canonical form a *doubly infinite* whirl (resp. curl), that is, a product of whirls (resp. curls) infinite in both directions?

From Section 10.

Problem 11.12. Let $X = \prod_{i=1}^\infty N(\mathbf{a}^{(i)})$ be an infinite product of curls, and suppose the given factorization of X is obtained from the ASW factorization by the action of $w \in S_\infty$ (via the maps η_i in Corollary 6.4). Find simple expressions for $a_j^{(i)}$ in terms minor ratio limits.

A special case of the following problem is discussed in [21].

Problem 11.13. Give an interpretation of minor ratio limits when both column and row indices are increasing sequences. When do such limits exist?

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